Input-Driven Pushdown Automata with Limited Nondeterminism

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August 27, 2014

Outline of the talk

- Input-driven PDAs: historical background and definitions

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    - Does a given NIDPDA have the k-path property?
- Open problems and further topics

Preliminaries

- A *pushdown automaton* reads input left-to-right and has access to finite-state memory and a pushdown stack.
- Each operation either reads a symbol from the stack (pop), pushes a string to the top of the stack (push) or does not change the stack. (Additionally, a PDA may have ε-transitions.)
- *Input-driven computation:* the input symbol determines whether the machines pushes or pops the stack, or does not touch the stack.

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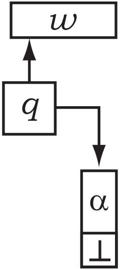
- *Next we'll define IDPDA computations and after that will summarize basic IDPDA decision properties.*

Definition: Input-driven pushdown automata

- $\Sigma = \Sigma_{+1} \cup \Sigma_0 \cup \Sigma_{-1}$: input alphabet;



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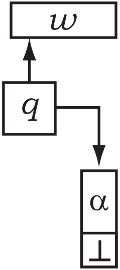

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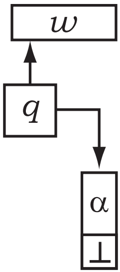

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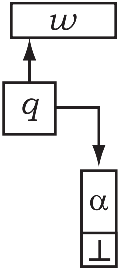

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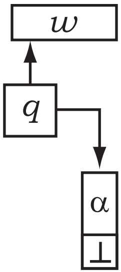

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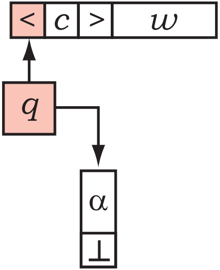

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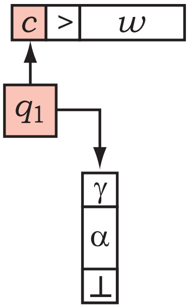

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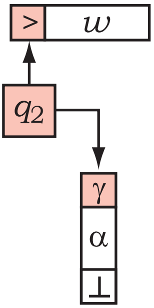

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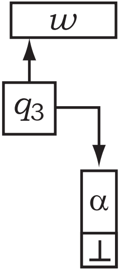
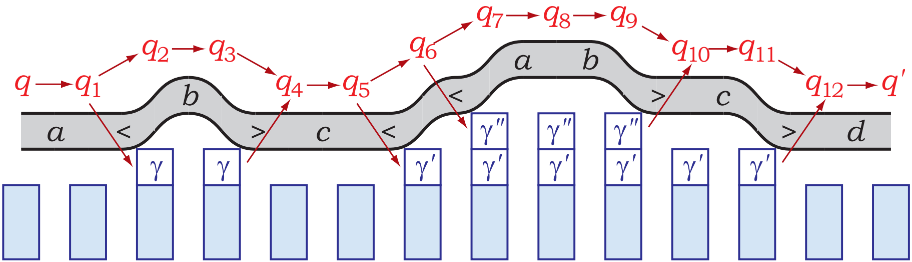
- $F \subseteq Q$: accepting states.



Computation of an IDPDA



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  - Closure under most standard language operations.
- Much ongoing research – motivated by new applications that use data with linear/hierarchical structure

Closure properties of central language families

	$\cup$	$\cap$	Complement	Concatenation	Kleene-*
Regular	Yes	Yes	Yes	Yes	Yes
CFL	Yes	No	No	Yes	Yes
DCFL	No	No	Yes	No	No
IDPDA	Yes	Yes	Yes	Yes	Yes

(D)CFL = (deterministic) context-free languages

Summary of decision properties

	Membership		Properties of a language		
	fixed	uniform	emptiness	equality	inclusion
DFA	regular	L	NL	NL	NL
NFA	regular	NL	NL	PSPACE	PSPACE
DIDPDA	in NC^1	in SC^2	P	P	P
NIDPDA	in NC^1	in P	P	EXPTIME	EXPTIME
DPDA	in $\mathrm{NC}^2 \cap \mathrm{SC}^2$	P	P	decidable	co-r.e.
CF	in NC^2	P	P	co-r.e.	co-r.e.

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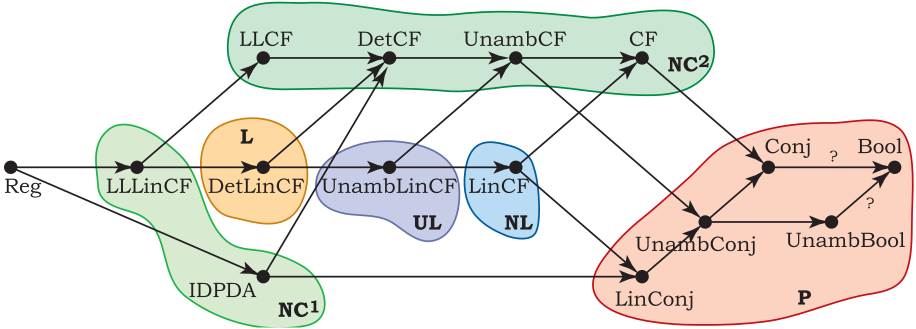
- Recent comprehensive survey:
  A. Okhotin, K. Salomaa, *Complexity of Input-Driven Pushdown Automata*, SIGACT News Complexity Theory Column 82 (Lane A. Hemaspaandra, Ed.), vol. 45, no. 2, June 2014, pp. 46–67

The big picture: IDPDAs among formal grammars



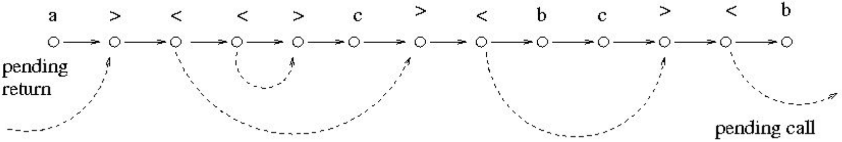
Equivalent models: pushdown forest automata

- *Pushdown forest automaton* (Neumann & Seidl 1998)
  - traverses input tree in depth-first left-to-right order
  - machine pushes onto the stack when going down to the leftmost child
  - pops from the stack when returning from the rightmost child
- Equivalent to IDPDA (Gauwin, Niehren & Roos, 2008)
- Recognize only the class of regular tree languages. Are *exponentially more succinct* than ordinary bottom-up tree automata
- Earlier related work:
  - Engelfriet, Rozenberg & Slutzki (1980): tree-walking transducers with synchronized pushdown
  - Kamimura & Slutzki (1981): nondeterministic and deterministic variants of such graph walking automata with a synchronized pushdown are equivalent

Equivalent models: Nested word automata

(Alur and Madhusudan, DLT 2006)

- A *nested word* is a tagged word with a hierarchical structure that connects call symbol occurrences to return symbol occurrences



- A *nested word automaton* “sends” finite state information both along the linear and the hierarchical edges
  - Equivalent to an IDPDA

Why nested words?

- Nested word automata used e.g. in XML document processing and model checking
  - Retains many desirable properties of the classical regular languages
- Advantages over trees in applications like document processing:
  - Word operations like prefix, suffix and concatenation do not have clear analogoues as tree operations
  - Trees do not have an *explicit* linear ordering of all nodes
    - Descriptional complexity: for tree automata queries that refer to the global linear order can be more expensive

In the following we use the terminology associated with IDPDAs. The model is equivalent to a finite automaton operating on nested words.

Determinizing nondeterministic IDPDAs (NIDPDA)

Theorem (von Braunmühl & Verbeek (1983), Alur & Madhusudan (2006))

Every NIDPDA of size n has a deterministic IDPDA of size $2^{O(n^2)}$.

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  - *Exponential size alphabet* can reach 2^{n^2}

Determinizing NIDPDAs: the number of stack symbols

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Theorem (Okhotin, Piao & Salomaa, 2012)

For all $k, h \in \mathbb{N}$, $k \leq h$, there exists a language $L_{k,h}$ recognized by an NIDPDA with $O(h)$ states and $O(k)$ stack symbols such that any DIDPDA for $L_{k,h}$ needs $\Omega(2^{k \cdot h})$ states and $\Omega(2^{k^2})$ stack symbols.

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- Tight bound with respect to both the number of states and the number of stack symbols (within a constant factor)

The lower bound due to Alur and Madhusudan

Lower bound: Let $\Sigma_{+1} = \{<\}$, $\Sigma_0 = \{0, 1, \#\}$, $\Sigma_{-1} = \{>\}$, consider all

$$< \ldots uv \ldots v > u$$

with $u, v \in \{0, 1\}^{\log n}$. (markers $\#$ omitted here and later)

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Upper bound: Remember sets of pairs (q, q') on each level of brackets:

$$\ldots < \underbrace{\ldots}_{q \to q'} > \ldots$$

Unambiguous Nondeterminism

Commonly used form of limited nondeterminism

- "If a string is accepted, it has a unique accepting computation".

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- In complexity theory:

$$L \subseteq UL \subseteq NL \qquad P \subseteq UP \subseteq NP$$

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Theorem (Okhotin, Salomaa 2011)

The worst-case UIDPDA–DIDPDA and NIDPDA–UIDPDA trade-offs are both $2^{\Theta(n^2)}$.

UIDPDA to DIDPDA: $2^{\Theta(n^2)}$

- Upper bound: known for NIDPDA $\to$ DIDPDA.

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with $u, v \in \{0, 1\}^{\log n}$, where the $(v)_2$-th bit in $x_{(u)_2}$ is 1.

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- IDPDA has to remember $x_0, \dots, x_{n-1}$.

NIDPDA to an unambiguous IDPDA

Lower bound method for unambiguous IDPDAs

Lemma (Schmidt, 1978)

Let $L \subseteq \Sigma^*$ *and* $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ *with* $x_i, y_i \in \Sigma^*$.
Define $M \in \mathbb{Z}^{n \times n}$ *by* $M_{i,j} = 1$ *if* $x_i y_j \in L$, *and* $M_{i,j} = 0$ *otherwise.*
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- To compute the rank, need a very simple matrix.

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- *Formal definitions in the proceedings.*

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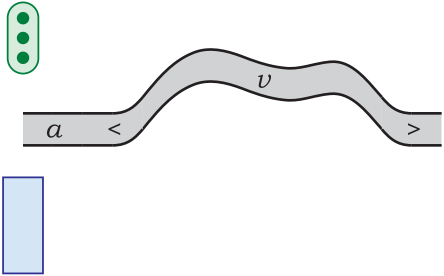

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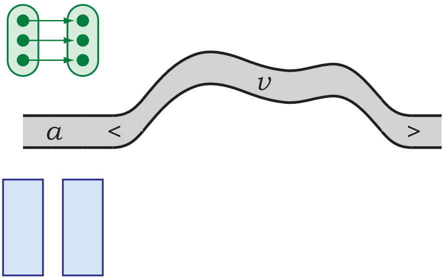

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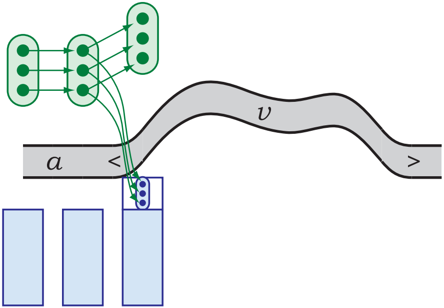

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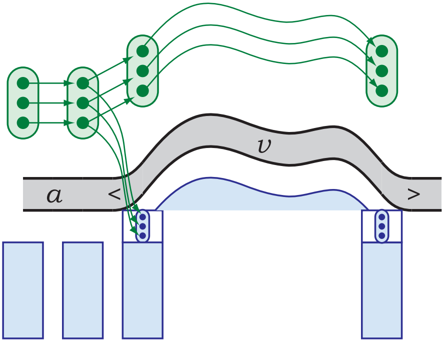

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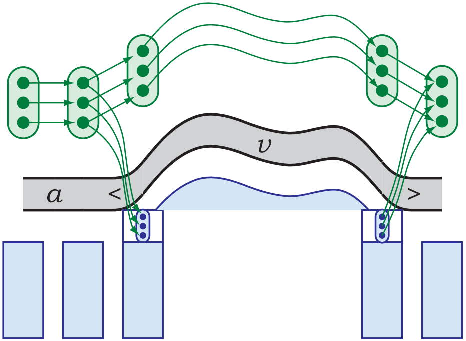

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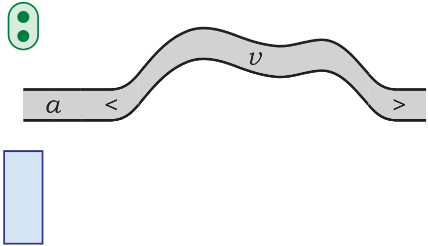

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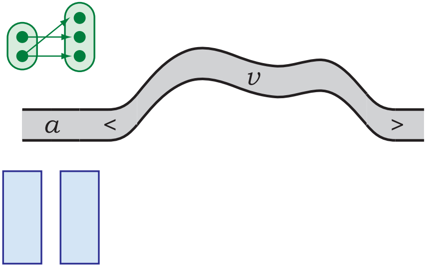

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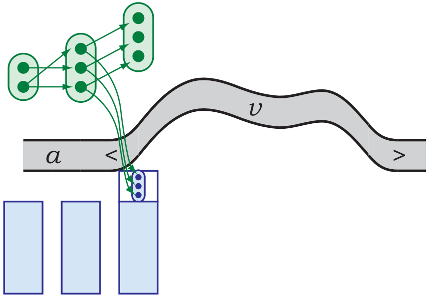

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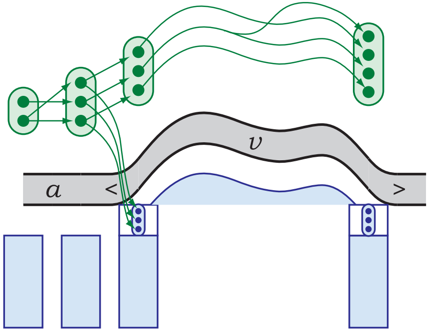

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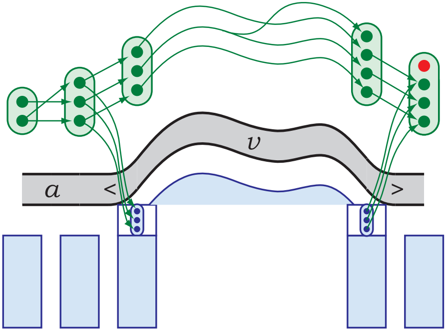

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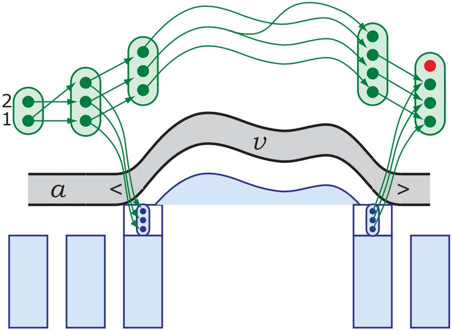

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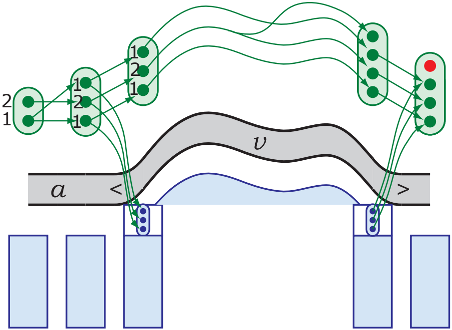

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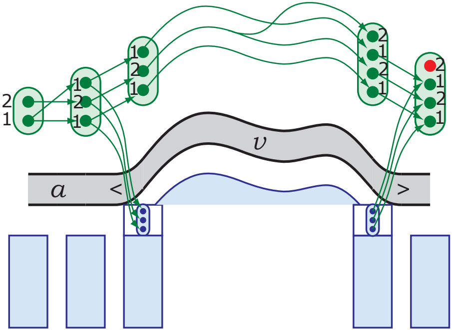

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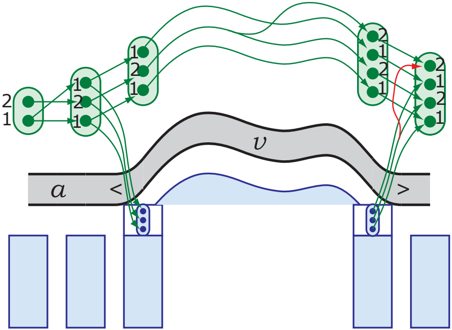

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- Use these data to match ℓ symbols to ℓ' states.

Lower bound on size blow-up of determinization

- For k-entry DIDPDAs, tight lower bound.
- The alphabet depends on n and k.

Lemma

For every $k \geqslant 1$ and $n \geqslant k$, there exists an alphabet $\Sigma^{k,n}$ and a language $L_{k,n}$ over $\Sigma^{k,n}$ recognized by a k-entry DIDPDA with n states and k stack symbols, such that any DIDPDA for $L_{k,n}$ needs $(n+1)^k - 1$ states.

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- the alphabet has one left bracket $<$, one right bracket $>$, and a large number of neutral symbols $\Sigma_0^{k,n} = X_{\text{func}} \cup Y_{\text{func}}$, where
  - $X_{\text{func}} = \{ \ a_f \mid f \colon \{1, \ldots, k\} \to \{1, \ldots, n, \text{undefined}\} \ \}$
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Lower bound language

(proof continued)

$$\widehat{L}_{k,n} = \{<a_f b_g> \mid a_f, b_g \in \Sigma_0^{k,n},\ \exists s \in \{1,\ldots,k\} :\ g(f(s)) = s\}.$$

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  - a k-entry DIDPDA for $\widehat{L}_{k,n}$ would need more states

Separator sets

(somewhat simplified definition)

A set $S = \{ \lhd x_1, \ldots \lhd x_m \}$, $x_1, \ldots, x_m \in \Sigma_0^*$, is a 1-separator set for language L if

- each $\lhd x_i$ is a prefix of some string in L,
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k-entry DIDPDA to DIDPDA

- For k-entry DIDPDAs the bound is tight

Theorem

- *A k-entry DIDPDA with n states and m stack symbols can be simulated by a DIDPDA with* $(n+1)^k - 1$ *states and* m^k *stack symbols.*

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- For 6-symbol alphabet: lower bound $(\frac{n}{4})^k$ on the size of a DIDPDA simulating a k-entry automaton with n states and k stack symbols.

k-path NIDPDA to DIDPDA

For a k-path NIDPDA with n states and m stack symbols:

- *upper bound:*

$$\sum_{i=1}^{k}(n+1)^i \cdot i^i \text{ states and } \sum_{i=1}^{k} m^i \text{ stack symbols.}$$

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- *lower bound:* $(n+1)^k - 1$ states
- The lower bound uses a k-entry DIDPDA (discussed above).
- *Open questions:*
  - Can the lower bound be improved for general k-entry DIDPDA?
  - Can the upper bound construction be improved?

NIDPDA to a k-path NIDPDA

Theorem

For any $k \geq 1$, the worst-case number of states in a k-path NIDPDA equivalent to an NIDPDA with n states is $2^{\Theta(n^2)}$.

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- Proof uses reduction from DIDPDA emptiness problem (in logspace)

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Open problem

Is it decidable whether or not a given NIDPDA has the finite path property?

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  Questions:
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