From algebra to logic: There and back again the story of a hierarchy

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- Algebra useful for classification (variety theory, Eilenberg *et al.*) and for decision algorithms
- FLT later also became useful in algebra and more ingredients came into FLT, especially topology: this will not appear today
- The hierarchy of langages discussed today, is an illustration of this interaction between FLT, algebra and logic

Logic and languages: which logic?

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- As it turns out, many important properties of complex systems can be specified in FO
- hence the interest in understanding the expressive power, the algorithmic properties, etc… of FO and some of its fragments

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- Note: deciding whether an automaton accepts an FO-definable language is PSPACE-complete (Cho, Huynh, 1991)

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- PTIME decision given a deterministic automaton

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- The product $(b^* a^+) b A^*$ is deterministic but not visibly deterministic

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- Let $\mathcal{R}_1 = \mathcal{L}_1 =$ piecewise testable languages, and let $\mathcal{R}_{m+1} = \mathcal{L}_m^{det}$, $\mathcal{L}_{m+1} = \mathcal{R}_m^{co-det}$ — where $\mathcal{V}^{det}$ is the class of all unions and intersections of deterministic products of languages in $\mathcal{V}$

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- $L = (b,c)^* bc^+ aA^* \in \mathcal{R}_3$ since $L = L_1 aA^*$, with $L_1 = (b,c)^* bc^+$ is a (visibly) deterministic product
- and $L_1 = (b,c)^* bc^+ = (b,c)^* bL_2$, with $L_2 = c^+$ is a (visibly) co-deterministic product and L_2 is piecewise testable

Decidability of membership problems in $\mathcal{R}_m$, $\mathcal{L}_m$

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- Moreover, if $\mathbf{V}$ (resp. $\mathbf{V}^{det}$) is the class of finite monoids corresponding to $\mathcal{V}$ (resp. $\mathcal{V}^{det}$), $\mathbf{V}^{det} = \mathbf{K} ⓜ \mathbf{V}$, that is: a monoid M is in $\mathbf{V}^{det}$ if and only if there exists a morphism $\varphi\colon M \to N$ with $N \in \mathbf{V}$ and for every idempotent $e \in N$, $\varphi^{-1}(e) \in \mathbf{K}$ (satisfies $x^\omega y = x^\omega$; is such that if $f = f^2$ and $s \in \varphi^{-1}(e)$, then $fs = f$)

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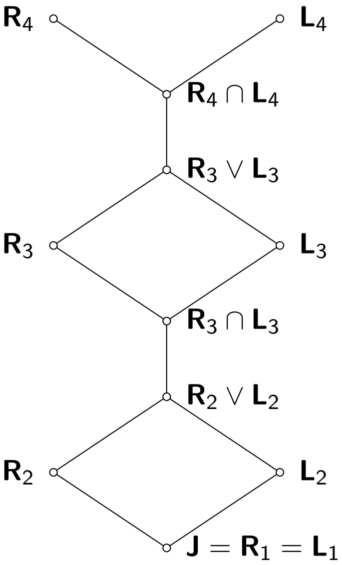
- The classes $\mathcal{R}_m$ and $\mathcal{L}_m$ are varieties of languages with decidable membership

The lattice generated by the $\mathcal{R}_m$ and $\mathcal{L}_m$



Second level: $\mathbf{R}_2 = \mathcal{R}$-trivial and $\mathbf{L}_2 = \mathcal{L}$-trivial monoids

joins and meets don't agree

$\mathbf{R}_m$ and $\mathbf{L}_m$ decidable

Infinite hierarchy?

What is its union?

Where **DA** and **B** come into the picture

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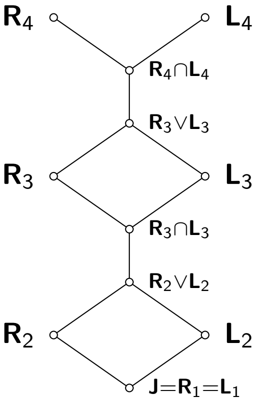
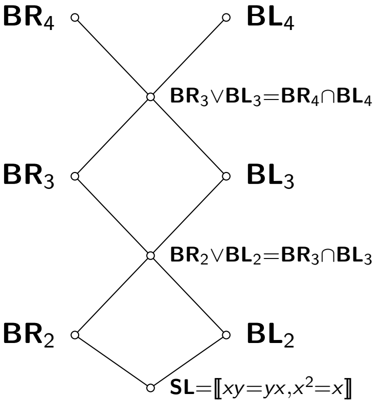
- So $\mathbf{R}_m, \mathbf{L}_m \subseteq \mathbf{DA}$
- The lattice of the $\mathbf{R}_m, \mathbf{L}_m$ resembles the lattice of idempotent monoid varieties, i.e. the sub-varieties of $\mathbf{B} = [\![x^2 = x]\!]$ — completely described in the 1970s and 1980s (Birjukov, Fennemore, Gerhard, Petrich)

Two similar lattices





where $\mathbf{BR}_{m+1} = (\mathbf{K} \cap \mathbf{B}) ⓜ \mathbf{BL}_m$ and $\mathbf{BL}_{m+1} = (\mathbf{D} \cap \mathbf{B}) ⓜ \mathbf{BR}_m$

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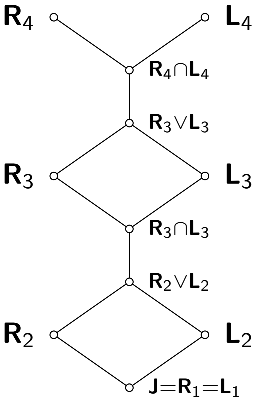

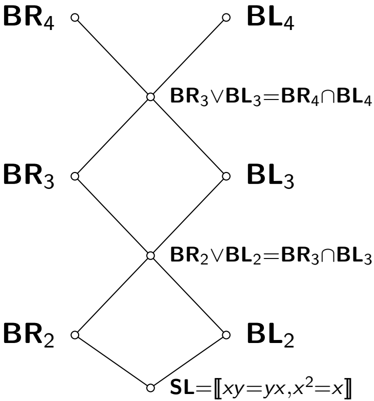

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The connection is provided by the map $\mathbf{V} \longmapsto \mathbf{V} \cap \mathbf{B}$, on the sub-varieties of $\mathbf{DA}$

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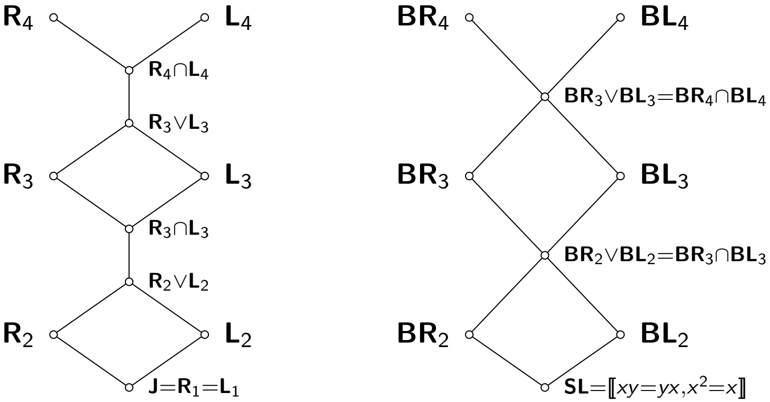

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(Trotter, W. 1997 + Kufleitner, W. 2012) $\mathbf{R}_m$ is maximal such that $\mathbf{R}_m \cap \mathbf{B} = \mathbf{BR}_m$ – and dually for $\mathbf{L}_m$

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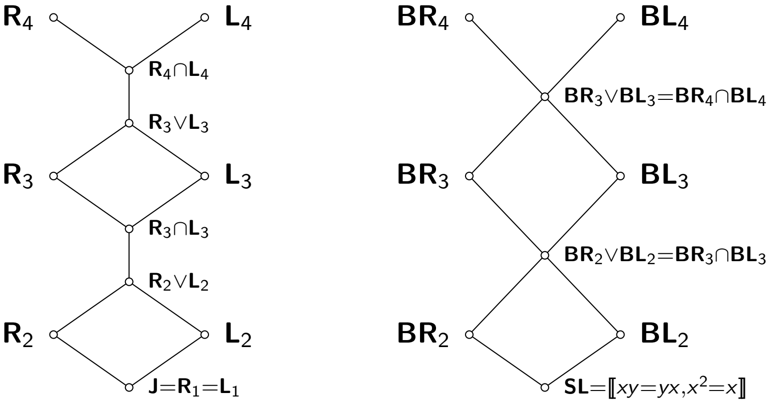

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The union of the $\mathbf{R}_m$, $\mathbf{L}_m$ is **DA** and the hierarchy is infinite – but finite on each finite alphabet

Exploiting our knowledge of idempotent monoids (from the 1970s)

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- A beautiful solution of the word problem considers, for a word w:

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- The word problem: given $u, v \in A^*$, are they equal in $FB(A)$ (written $u =_{FB} v$)? Ex: $abacbcacbacabca =_{FB} abacbca$
- A beautiful solution of the word problem considers, for a word w:
- $w = s(w)\sigma(w)w' = w''\tau(w)t(w)$ where $\sigma(w)$ is the last letter to first appear (from the left), $s(w)$ is the corresponding prefix, and $t(w)$, $\tau(w)$ are the left-right duals.

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- That is: find the last letter to have its first occurrence in u from the left (resp. the right), then doing the same in the maximal prefixes and suffixes with alphabetical content one less letter, etc.

Rankers and the varieties $\mathbf{R}_m$ and $\mathbf{L}_m$ 1/3

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- Dual definition for $\underline{R}^{Y}_{m,n}$
- The property "the same rankers from $\underline{R}^{X}_{m,n}$ are well-defined on u and v" (written $u \rhd_{m,n} v$) is a finite index congruence. Dually for $\underline{R}^{Y}_{m,n}$, written $u \lhd_{m,n} v$

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- This yields a new characterization of the $\mathbf{R}_m$ and $\mathbf{L}_m$ (Kufleitner, W. 2012)
- **Theorem** $\mathbf{R}_m$ is the variety of monoids generated by the $A^*/\triangleright_{m,n}$ $(n \geq m)$, and $\mathbf{L}_m$ is generated by the $A^*/\triangleleft_{m,n}$ $(n \geq m)$. Moreover, $\mathcal{R}_{m+1} = \mathcal{L}_m^{v-det}$ and $\mathcal{L}_{m+1} = \mathcal{R}_m^{v-det}$

Back to logic: rankers and fragments of first-order logic

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- *Limitation of resources*: a formula is more or less *complex* if it requires a larger or smaller number of nested quantifiers; the quantifier alternation hierarchy
- *Limitation of resources*: a formula may use more or less variable symbols; at the cost of making the formula longer (Adler & Immerman 2002, Grohe & Schweikardt 2005)

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$$A^* a_1 A^* a_2 A^* a_3 A^* a_4 A^*$$

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First order logic with two variables

- FO^2 is properly contained in $\mathbf{Ap} = FO = FO^3$
- FO^2 is a variety of languages: a language is FO^2-definable if its syntactic monoid is in **DA** (Thérien, Wilke, 1998)

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- **Questions**: Is the quantifier alternation hierarchy infinite? Can we characterize its levels? Can we decide membership in its levels?
- A collection of results of Schwentick, Thérien, Vollmer (2001); Weis, Immermann (2007,2009); Kufleitner, Weil (2009, 2012); Straubing and Krebs (2011, 2012)

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- (combinatorially highly non trivial)

The final result

Theorem

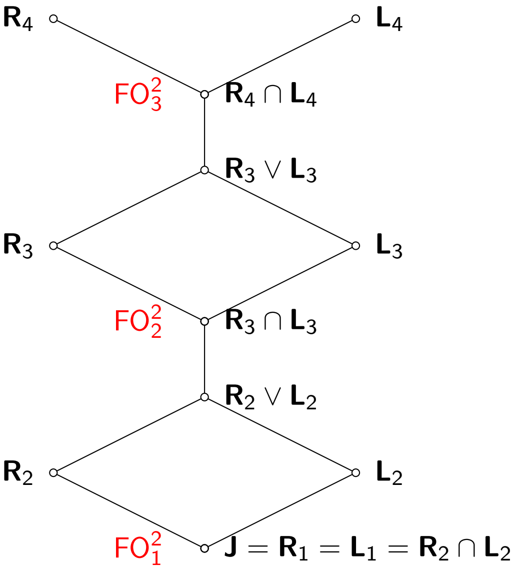
$FO^2_m = \mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$

FO^2_m is decidable

the hierarchy collapses

at level $m + 1$

on an m-letter alphabet



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- **Theorem** (Straubing, 2011) For each m, $\mathbf{V}_m = \mathrm{FO}^2_m$
- **Corollary** (Straubing and Krebs, 2012) Each FO^2_m is decidable

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  $g_{m+1} = x_{m+1}\overline{g_m}, \quad h_{m+1} = g_{m+1}x_{m+1}\overline{h_m}$
  $\varphi\colon x_1 \mapsto (x_1^\omega x_2^\omega x_1^\omega)^\omega \;\; x_2 \mapsto x_2^\omega \;\; x_{m+1} \mapsto \left(x_{m+1}^\omega \varphi(g_m\overline{g_m})^\omega x_{m+1}^\omega\right)^\omega$

Thank you for your attention!