Synchronizing Automata with Random Inputs

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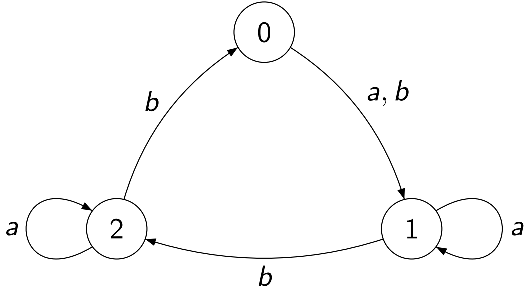
We consider DFA: $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$

Any w with this property is a synchronizing word for $\mathscr{A}$

The minimum length of synchronizing words for $\mathscr{A}$ is called the reset threshold of $\mathscr{A}$ and denoted by $\mathrm{rt}(\mathscr{A})$

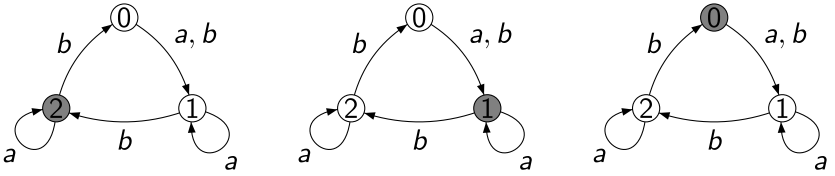


The word *abba* is synchronizing

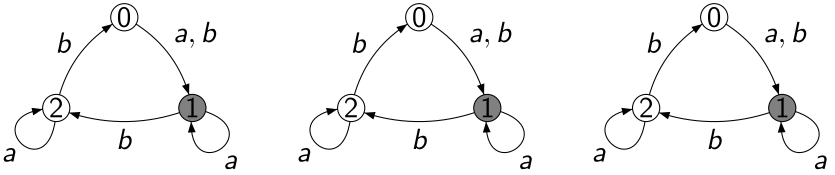
It resets the automaton to the state 1

Classical synchronization problem:

States of objects are unknown, but we control environment



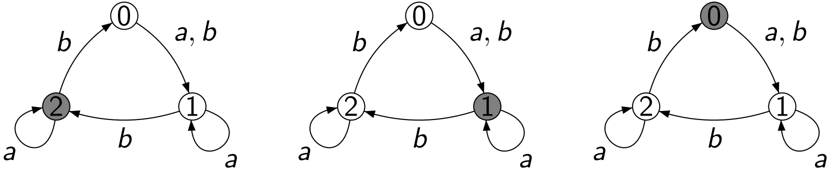
Environment: ⇓ *abba* ⇓



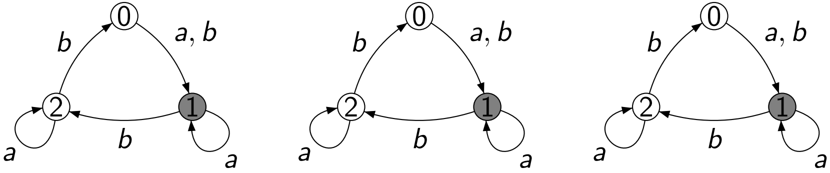
Finally, all objects are in the same state

Randomized synchronization problem:

We do not control environment. It is *random*.



Environment: ⇓ Sequence of random letters ⇓



How long do we have to wait until all objects are in the same state?

Let Σ be a binary alphabet $\{a, b\}$

Model for the environment: *Bernoulli process* $\mathcal{B}(p, q)$

Letters are drawn independently and sequentially

letter a with the probability p

letter b with the probability $q = 1 - p$

We are interested in the *expected number of letters* drawn from $\mathcal{B}(p, q)$ until an automaton $\mathscr{A}$ is synchronized

Is it finite? *Sure.*

Let w be a synchronizing word of an automaton $\mathscr{A}$.
If (ℓ_a, ℓ_b) is the Parikh vector of w then the expected number of letters until synchronization of $\mathscr{A}$ is at most $\frac{|w|}{p^{\ell_a} q^{\ell_b}}$

Let n be an odd integer, and $\Sigma = \{a, b\}$

$\mathscr{U}_n$ is the minimal automaton of the language $\Sigma^* a^{\frac{n+1}{2}} b^{\frac{n-1}{2}} \Sigma^*$

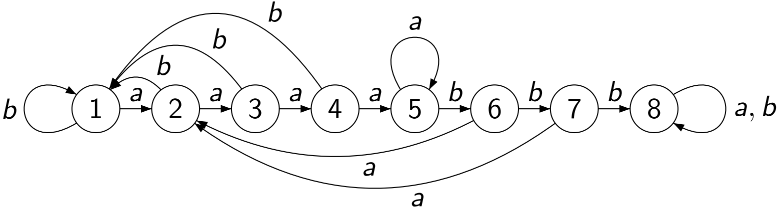


Figure: Automaton $\mathscr{U}_7$

Lemma

A word w is synchronizing if and only if it brings the state 1 to the sink state.

Note, the reset threshold of $\mathscr{U}_n$ is equal to $n - 1$

Synchronization in a random environment is equivalent to a random walk from the state 1 until absorption in the sink state

Recall, $\mathscr{P}(a) = p$ and $\mathscr{P}(b) = q$

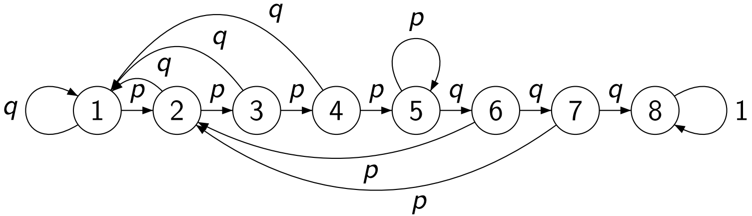


Figure: Markov chain for $\mathscr{U}_7$

Let μ_i be the expected number of steps until absorption in the sink state

$$\begin{cases} \mu_8 = 0 \\ \mu_7 = p\mu_2 + q\mu_8 + 1 \\ \mu_6 = p\mu_2 + q\mu_7 + 1 \\ \dots \end{cases}$$

System of linear equations in the general case:

$$\begin{cases} \mu_1 = p\mu_2 + q\mu_1 + 1 & (1) \\ \mu_i = p\mu_{i+1} + q\mu_1 + 1, \text{ if } 1 \leq i \leq \frac{n+1}{2} & (2) \\ \mu_{\frac{n+3}{2}} = p\mu_{\frac{n+3}{2}} + q\mu_{\frac{n+5}{2}} + 1 & (3) \\ \mu_i = p\mu_2 + q\mu_{i+1} + 1, \text{ if } \frac{n+5}{2} \leq i \leq n-1 & (4) \\ \mu_n = p\mu_2 + q\mu_{n+1} + 1 & (5) \\ \mu_{n+1} = 0 & (6) \end{cases}$$

Theorem

The expected number of letters, that are drawn from $\mathcal{B}(p, q)$, until $\mathscr{U}_n$ is synchronized, is equal to $\frac{1}{p^{\frac{n+1}{2}} q^{\frac{n-1}{2}}}$ if n is odd, and is equal to $\frac{1}{p^{\frac{n}{2}} q^{\frac{n}{2}}}$ if n is even.

The Černý automaton $\mathscr{C}_n$ has the largest known reset threshold

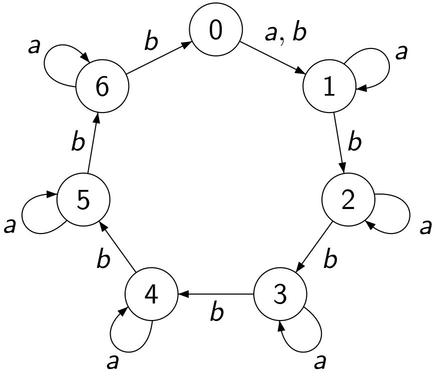


Figure: The automaton $\mathscr{C}_7$

We have to deal with pairs

A pair $\{s, t\}$ is synchronized by w if $\delta(s, w) = \delta(t, w)$

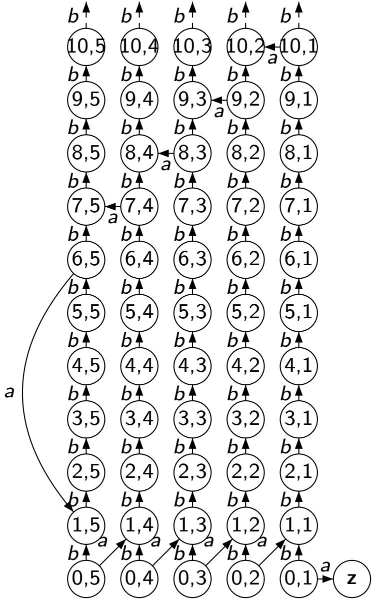


Figure: Pair automaton of $\mathscr{C}_{11}$

Theorem (odd)

Let n be a positive odd integer. The expected number of letters, that are drawn from $\mathcal{B}(p,q)$, until the pair $\{1, \frac{n-1}{2}\}$ of $\mathscr{C}_n$ is synchronized, is equal to $\frac{(n-1)((n-1)^2+q(3n-5)+4q^2)}{8pq^2}$.

Theorem (even)

Let n be a positive even integer. The expected number of letters, that are drawn from $\mathcal{B}(p,q)$, until the pair $\{1, \frac{n}{2}\}$ of $\mathscr{C}_n$ is synchronized, is equal to $\frac{n((n-1)(n-2)+q(3n-6)+4q^2)}{8pq^2}$.

Expected number of letters until $\mathscr{C}_n$ is synchronized is at most $\frac{(n-1)^4}{8pq^2}$

Outline :

Notion of randomized synchronization of automata

Automaton $\mathscr{U}_n$ is easy to synchronize in the classical setting ($n-1$ letters) and is hard to synchronize in the random setting (exponential number).

Opposite situation for the Černý automata

Future work :

Clear picture of this notion

Similar notion for a two-sided ideal language $\mathscr{L}$:

the expected number of letters you need to draw until you get a word from the language $\mathscr{L}$