On k-abelian palindromic rich and poor words

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Framework

Palindromic words

k-abelian equivalence

k-abelian equivalence

Definition

For $k \geq 1$, words u and v in Σ^+ are *k-abelian equivalent*

$$u \equiv_k v$$

iff

for each $l \leq k$ and $w \in \Sigma^l$: $|u|_w = |v|_w$.

Obvious facts:

- $k = 1$ is just abelian equivalence;
- $k = \infty$ is the equality of words;
- for $k' < k$, $u \equiv_k v \Longrightarrow u \equiv_{k'} v$
- $u = v \Longleftrightarrow \forall k \quad u \equiv_k v$, i.e.,

$$= \quad \equiv \quad \bigcap_{k \geq 1} \equiv_k$$

- k-abelian equivalence is a congruence

Equivalently,

Definition

$$u \equiv_k v$$

iff

- $\forall w \in \Sigma^k$: $|u|_w = |v|_w$
- $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$ or
- $u = v$ for words $u, v \in \Sigma^{\leq k}$

A hierarchy of approximations of the equality

$$= \quad \subseteq \quad \dots \quad \subseteq \quad \equiv_k \quad \subseteq \quad \equiv_{k-1} \quad \subseteq \quad \dots \quad \subseteq \quad \equiv_0$$

where $\equiv_0$ denotes the equal length relation

The number of the equivalence classes

$$F_{k,m}(n) = \Theta(n^{(m-1)m^{k-1}}),$$

where

$n =$ the length of the word
$m =$ the size of the alphabet
$k =$ the k-abelian constant

In particular,

- $F_{2,2}(n) = n^2 - n + 2$
- $F_{3,2}(n) = \Theta(n^4)$
- $F_{4,2}(n) = \Theta(n^8)$

Problems

Basic goal:

When does the k-abelian equivalence $\equiv_k$ behave like the equality $=$ and when like the abelian equality $\equiv_1$?

The problems considered here are

- decision problems (K., ICALP81)
- complexity (K., Saarela, Zamboni 2013, Cassaigne, K., Saarela 2014)
- unavoidability (Huova 2014, Rao 2013)
- palindromicity (here)

Decision problems (K, 1981)

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The following problems are decidable:

- k-abelian PCP:

$$\exists u, v : u \equiv_k v \& h(u) = g(v)?$$

- k-abelian (H)D0L-problem:

$$f_1 h^n(w) \equiv_k f_2 g^n(w) \quad \forall n?$$

Problems as in the case of abelian equality.

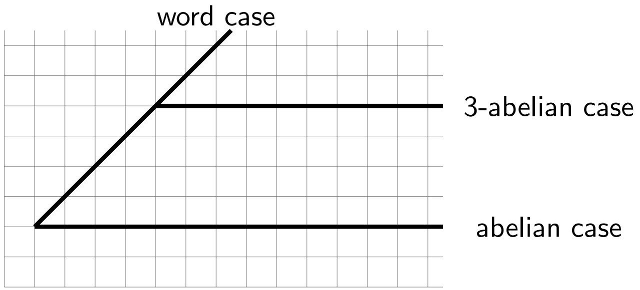
Small complexity

Definition

Complexity:

$$C_w(n) = \text{Card}\ \{[v] | v \in F(w) \bigcap \Sigma^n\}$$

The lowest complexity of aperiodic words characterize the Sturmian words (K-Saarela-Zamboni, 2013)



Small complexity

Example

Thue-Morse word:

- $\lim \sup(n) = \Theta(\log(n))$
- $\lim \inf(n) = 6$

- k-abelian complexity can be heavily fluctuating
- jumps from level k to $k + 1$ can be "maximal"

Unavoidability

Avoiding squares and cubes of different types in infinite words:

Card(Σ)\	type of squares $=$	type of squares $\equiv_2$	type of squares $\equiv_1$	type of cubes $=$	type of cubes $\equiv_2$	type of cubes $\equiv_1$
2	$-$	$-$	$-$	$\oplus$	$\text{II}_?$	$-$
3	$\oplus$	$\text{I}_?$	$-$	$+$	$+$	$\oplus$
4	$+$	$+$	$\oplus$	$+$	$+$	$+$

$\oplus$: classical results

$+$ or $-$: direct consequences

$\text{I}_? \longleftrightarrow -$; the longest word of length 537, Huova, 2012

$\text{II}_? \longleftrightarrow +$; Rao, 2013

Palindromicity

Palindromicity with respect to

- words: $w = w^R$
- abelian case: any word is abelian palindrome
- k-abelian case: $w \equiv_k w^R$

How many palindromes a word can contain?
Central notions:

- rich words: maximal number of palindromes as factors
- poor words: minimal (or finite) number of palindromes as factors in an infinite word

Rich and poor words: known

Rich words:

- The maximal number of palindromes a word of length n can contain is $n+1$.
- Such words exist and are called palindrome rich (folklore)

Poor words:

- There exists a reversal closed infinite word containing only finitely many palindromes (Berstel-Boasson-Carton, 2009)
- the smallest number of palindromes such a word can contain is 14 (Fici-Zamboni, 2013)

Rich and poor words of different types

	$=$	$\equiv_k$	$\equiv_1$	
poor	C	C	∞	for infinite words
rich	$n+1$	$\Theta(n^2)$	$\Theta(n^2)$	for finite words

Rich and poor words of different types

	$=$	$\equiv_k$	$\equiv_1$	
poor	C BBC	C our result	∞ trivial	for infinite words
rich	$n+1$ folklore	$\Theta(n^2)$ our result	$\Theta(n^2)$ trivial	for finite words

- for poor words k-abelian equality behaves like equality
- for rich words k-abelian equality behaves like abelian equality

Our main result for poor words

Let k be the k-abelian constant and l the cardinality of the alphabet. The following table characterizes the existence of (k, l)-poor words:

$k \backslash l$	2	3	4	5
1	$-$	$-$	$-$	$-$
2	$-$	$\ominus$	$\oplus$	$+$
3	$-$	$\oplus$	$+$	$+$
4	$\ominus$	$+$	$+$	
5	$\oplus$	$+$		

The crucial things to prove are the marked $\oplus$ cases, as well as the marked $\ominus$ cases.

Proving existence

Ideas to prove marked $\oplus$ cases:

- we modify the construction of sesquipowers:

$$U_0 = w_0,$$
$$U_{i+1} = U_i w_i U_i^R,$$

  where w_i is a suitable marker
- a crucial technical thing is to define the markers so that only finitely many k-abelian palindromes are obtained.

We can find a bound for the finiteness but that would be large and not even close to the optimal value!

Proving nonexistence

Idea to prove the $\ominus$ cases (example for $k = 2$, $|\Sigma| = 3$):

- Rewriting rules which do not affect the 2-palindromicity:
  - for $x \in \Sigma$, substitute $xx \to x$
  - for $x, y \in \Sigma$, substitute $xyx \to x$
- reduce a word to a reduced form when no rewriting rule can be applied
- the reduced form of any ternary word v is a factor of $(abc)^\infty$ or $(cba)^\infty$
- a ternary word v is a 2-palindrome if and only if its reduced form is a letter
- using $|\Sigma| = 3$ and the reduced form, we find long palindromes

Our main result for k-abelian rich words

- Consider words of the form:

$$v = a^l (ba^{k-1})^m$$

- Let $|w| = n$ and $n \geq k$. We can count all k-abelian palindromes and maximaze their number by the choice of l and m:

$$\sharp \text{ of k-abelian palindromes in } w \geq 1/4kn^2$$

- These words are k-abelian rich: Since a word of length n contains $\Theta(n^2)$ factors in total, it cannot contain more than $\Theta(n^2)$ k-abelian palindromes. Our words do contain $\Theta(n^2)$ inequivalent abelian palindromes.

Open problems

uniformly recurrent poor word

Does there exist an infinite uniformly recurrent reversal closed word with only finitely many k-abelian palindromes?

optimal numbers of palindromes

What is the exact minimal number of k-abelian palindromes an infinite reversal closed word can contain? What is the exact maximal number of distinct k-abelian palindromes a word of length n can contain?

infinite k-abelian rich word

Does there exist an infinite k-abelian rich word? I.e., does there exist an infinite word w, such that for some constant C each of its factors of length n contains at least Cn^2 distinct k-abelian palindromes?

Thanks

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