Aperiodic tilings and entropy

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Wang tiles

Wang tiles

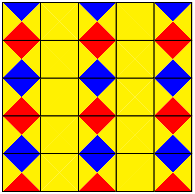
- Finite set of colors
- Alphabet = colored squares
- Adjacent borders have matching colors

$c = \{$red, yellow, blue$\}$

$\Sigma = \{$  $\}$

$\Sigma' = \{$  $\}$

. . .

. . .  . . .

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Aperiodic tilesets

Definition

A set of tiles is **aperiodic** when:

- it can cover the plane;
- it cannot cover the plane periodically.

Cover such that adjacent borders have matching colors

The history of small aperiodic tilesets

Self-similar

1964	R. Berger	$> 20,000$ tiles
1966	D. Knuth	96 tiles
1971	R. Robinson	52 tiles
1974	R. Penrose	32 tiles
1986	R. Ammann, B. Grünbaum, G. Shephard	16 tiles

Not self-similar

1996	J. Kari	14 tiles
1996	K. Culik	13 tiles

Our result

Theorem

The Kari-Culik tileset has positive entropy.

Intuitively:

- Description of a $n \times n$-square takes $\Omega(n^2)$ bits
- It contains dense “random” bits

Note: entropy zero was conjectured.

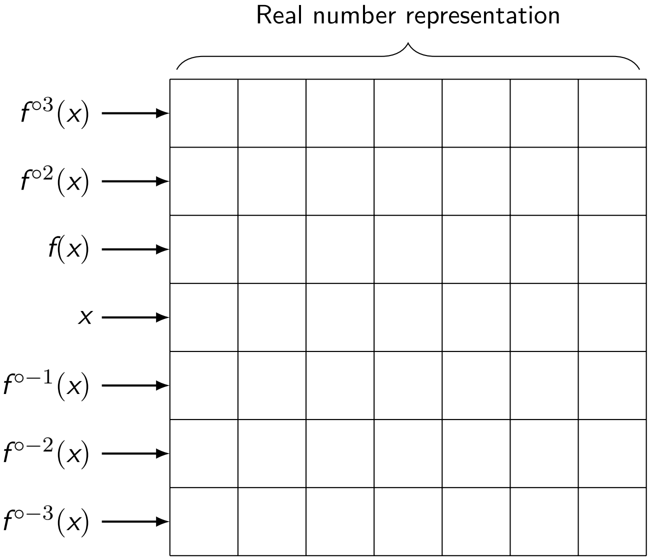
A function with aperiodic orbits

Consider this function:

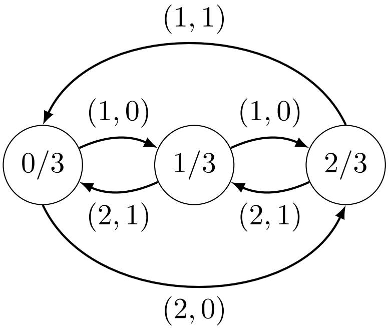
$$f \colon [\frac{1}{3}; 2] \to [\frac{1}{3}; 2]$$
$$x \mapsto \begin{cases} 2x & \text{if } x \leq 1 \\ x/3 & \text{if } x \geq 1 \end{cases}$$

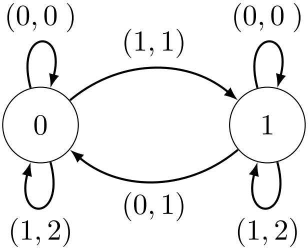
- Its orbits, i.e. sequences $u_x = (f^{\circ n}(x))_{n \in \mathbb{N}}$, are aperiodic
- Its orbits are also dense in $[\frac{1}{3}; 2]$

The general idea



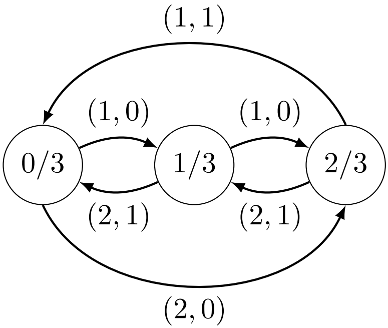
Multiplications done by transducers

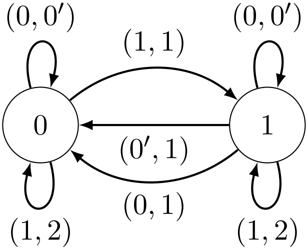




- $(2, 0)$ stands for "read 2, write 0"

Multiplications done by transducers





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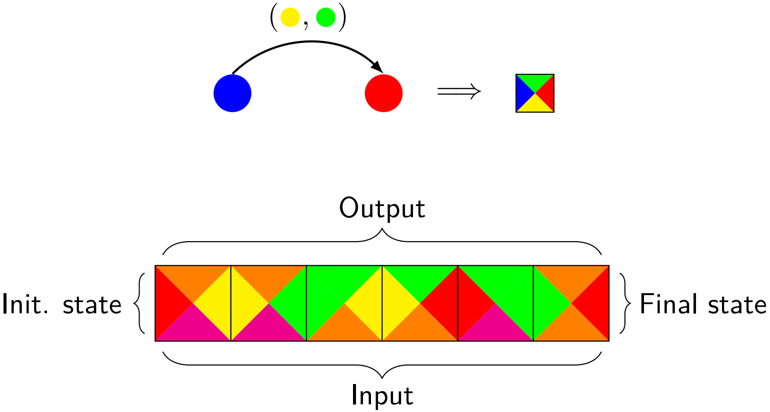
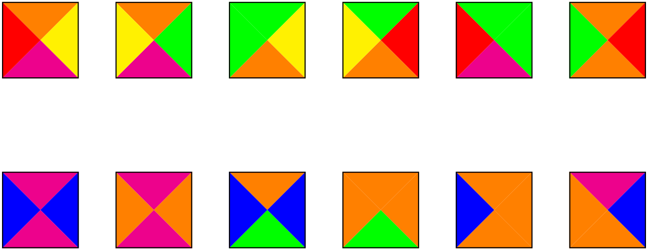
From transducers to tile sets

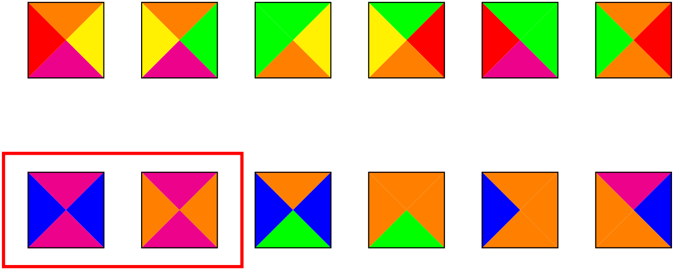


- States of $M_{1/3}$, M_2: disjoints colors
- One line = one run

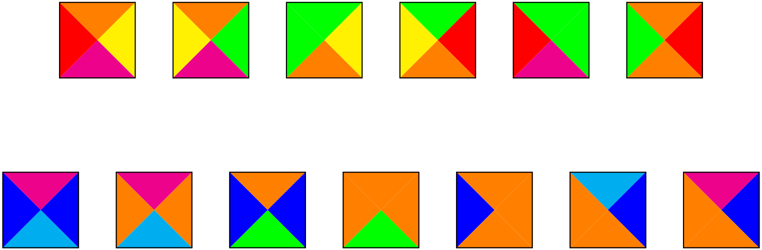
An aperiodic set of tiles



An aperiodic set of tiles



An aperiodic set of tiles



Lines and averages

Definition

The *average* of a sequence (u_n) is:

$$\mathrm{avg}(u) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} u_k$$

Tilings: average of a line = average of northern sides

Aperiodicity

Theorem (J. Kari and K. Culik, 1996)

The Kari-Culik tileset is aperiodic.

Sketch of proof.

- Suppose there is a periodic tiling. Then each line has an average. The averages are periodic: contradiction.
- To tile the plane, start from $\ldots 11111111 \ldots$ and run the transducers forever.

□

Encoding real numbers in bi-infinite sequences

$$S_x(n) = \lfloor (n+1)x \rfloor - \lfloor nx \rfloor$$

$$S_{1/2} = \dots 01\,01\,01\,01\,01\,01\,01\,01\,01 \dots$$
$$S_{7/5} = \dots 2112111\,2112111\,2112111 \dots$$
$$S_{\pi/3} = \dots 2111111111111111111112111111111111111111112 \dots$$

- S_x is on alphabet $\{\lfloor x \rfloor, \lceil x \rceil\}$
- The average of values of S_x is x

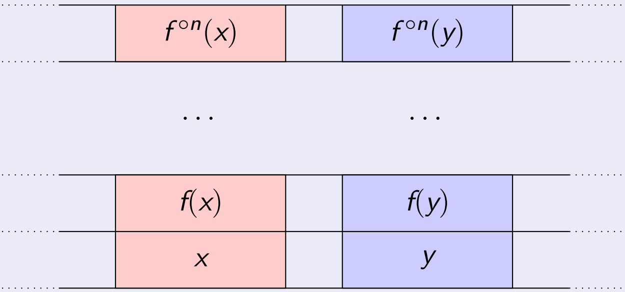
Each line has an average

Lemma

In any tiling, each line has an average.

Sketch of proof.

Consider a line without an average.



Density $\implies \exists n$ s.t. $f^{\circ n}(x) < 1 < f^{\circ n}(y)$ □

Entropy

$C(n)$ = the number of $n \times n$-squares found in any tiling

Definition

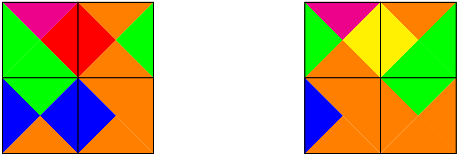
We call the **entropy** the following quantity:

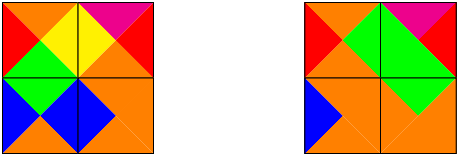
$$E = \lim_{n \to \infty} \frac{\log C(n)}{n^2}$$

- Classical definition in dynamical systems
- With 13 tiles, if $C(n) \sim 13^{\epsilon n^2}$, then $E = \epsilon$

Substitutive pairs

are pairs of distinct patterns with the same borders.

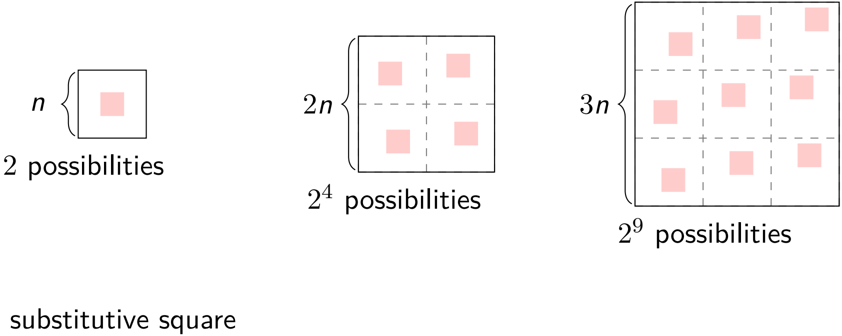




Substitutive pairs generate entropy

Lemma

If a substitutive square is found in any $n \times n$-square of any tiling, then the entropy of the tiles is positive.



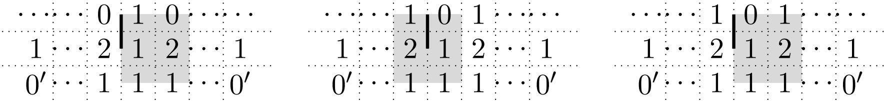
Substitutive pairs appear often (1/2)

Lemma

Whenever a pattern $0111^{\alpha}0$ *occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

Sketch of proof.

Case analysis. □



Middle case

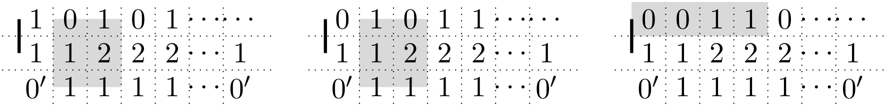
Substitutive pairs appear often (1/2)

Lemma

Whenever a pattern $0111^{\alpha}0$ *occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

Sketch of proof.

Case analysis. □



Leftmost case

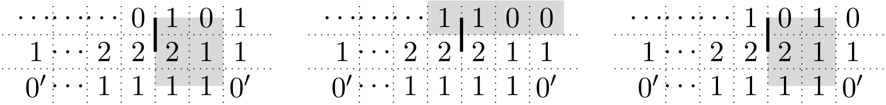
Substitutive pairs appear often (1/2)

Lemma

Whenever a pattern $0111^{\alpha}0$ *occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

Sketch of proof.

Case analysis. □



Rightmost case

Substitutive pairs appear often (2/2)

Lemma

In any line with an average $\in]\frac{3}{4};\frac{9}{10}[$*, a pattern of the form* $0111^{\alpha}0$ *appears regularly.*

Sketch of proof.

If there are no "0" regularly, then the average is 1.

If there are no "111" regularly, then the average is $\leq \frac{3}{4}$. □

- All orbits regularly meet the interval $]\frac{3}{4};\frac{9}{10}[$
- Hence substitutive squares appear often enough

Many thanks for your attention!

Our result

- The entropy of the Kari-Culik tileset is positive
- The Kari-Culik-tilings are not all self-similar

Open problems

- Characterize the language of words which can appear on K.C.'s lines?
- Is there a tileset working the same way, but with 0 entropy?
- Is there a sub-shift of finite type A, with positive entropy, such that any subshift of finite type $\subset$ A also has positive entropy?