Breadth-first signature of trees and rational languages

Victor Marsault,
joint work with Jacques Sakarovitch

CNRS / Telecom-ParisTech, Paris, France

Developments in Language Theory 2014, Ekateringburg,
2014–08–30

Breadth-first serialisation of languages and numeration systems:

The rational case

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Outline

We call *tree* a...

Directed graph which is

- **Rooted:** a node is called *the root* (leftmost in the figures)
- **Directed outward from the root:** there is a unique path from the root to every other node.
- **Ordered:** the children of every node are ordered
  (In the figures, lower children are smaller.)

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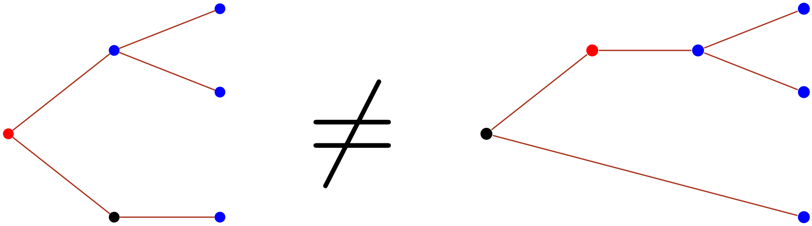

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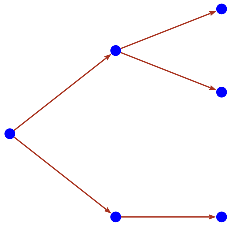

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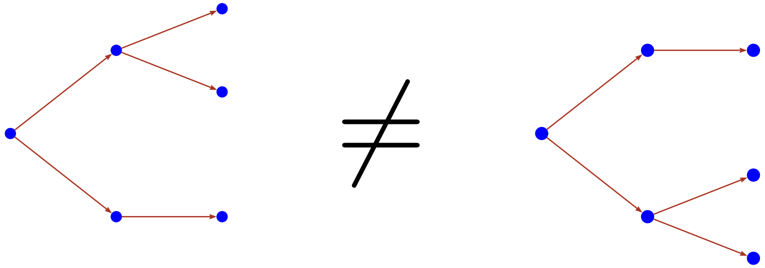
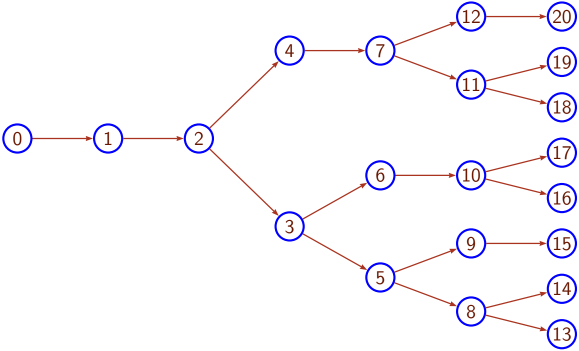


Every tree has a canonical breadth-first traversal



Two more features

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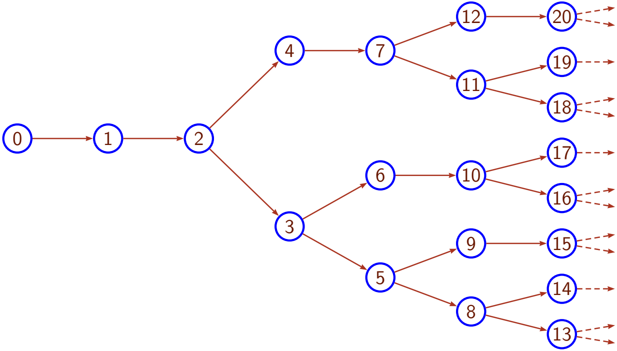
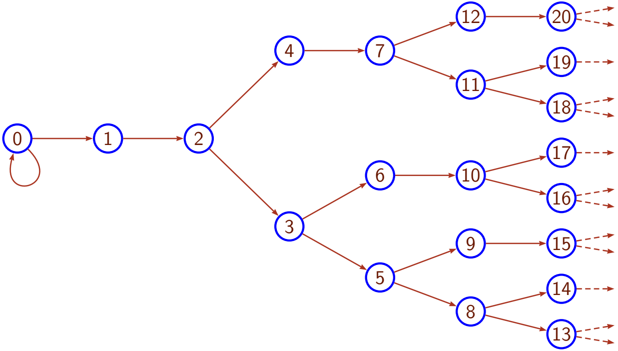


Two more features

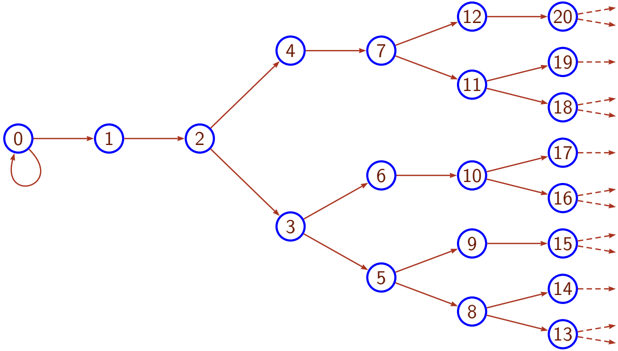
- We consider infinite trees only.
- For convenience, there is loop on the root.



Signature of a tree

Definition

The **signature** of a tree is the sequence of the degrees of the nodes taken in breadth-first order.



$\mathbf{s} =$

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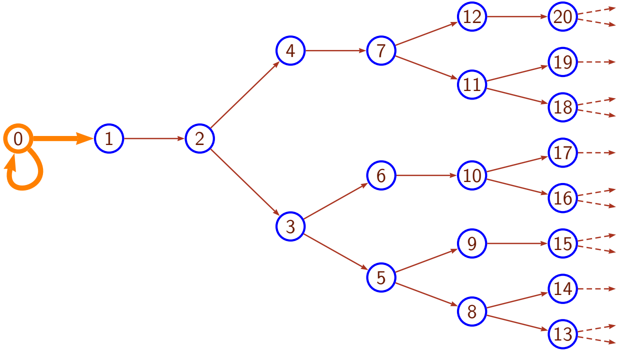

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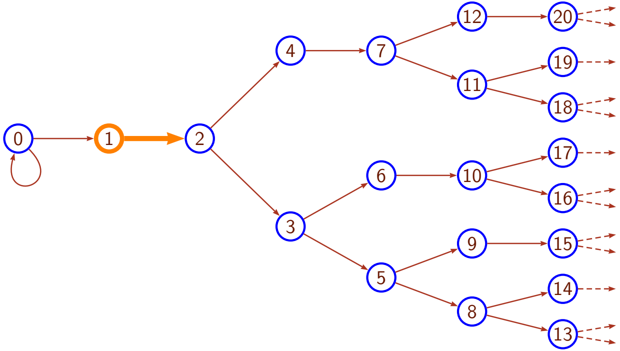

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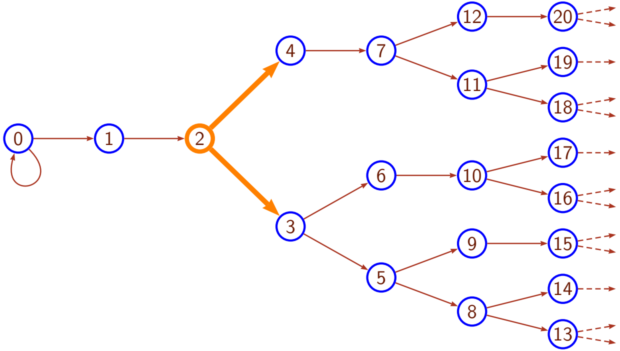

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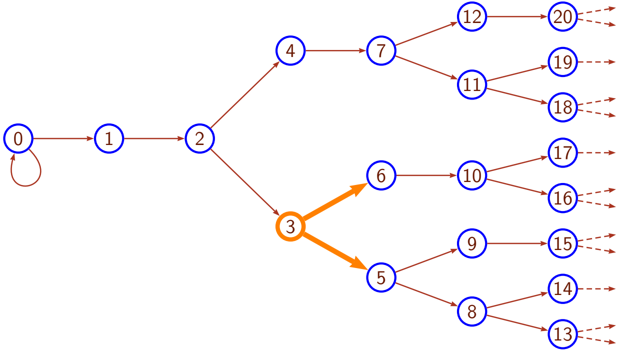

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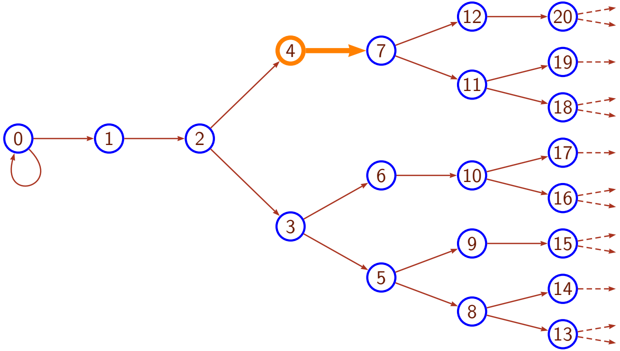

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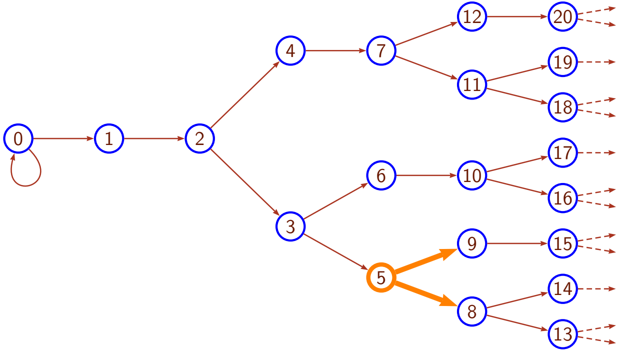

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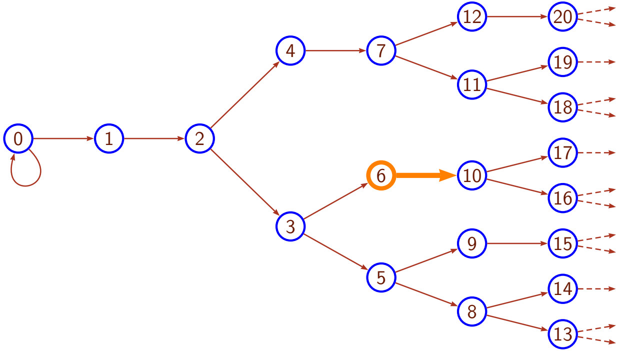

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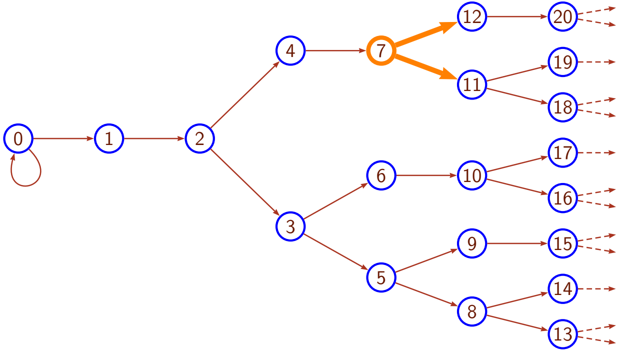

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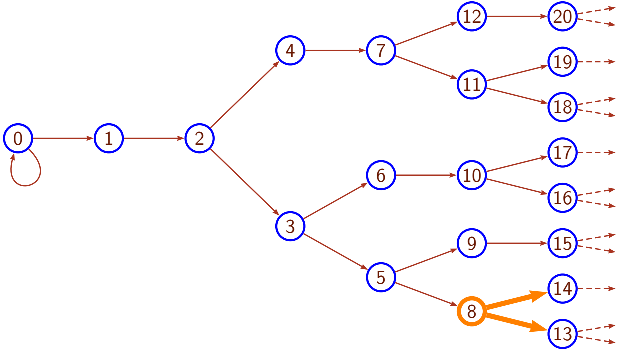

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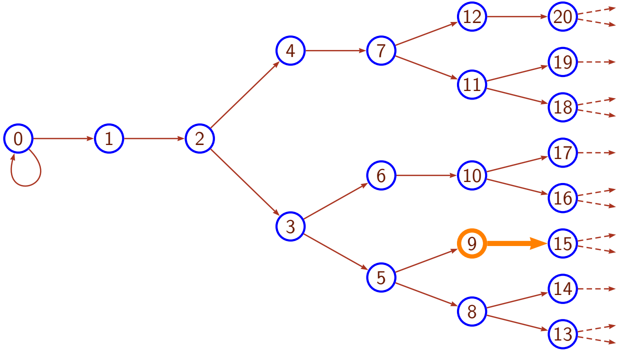

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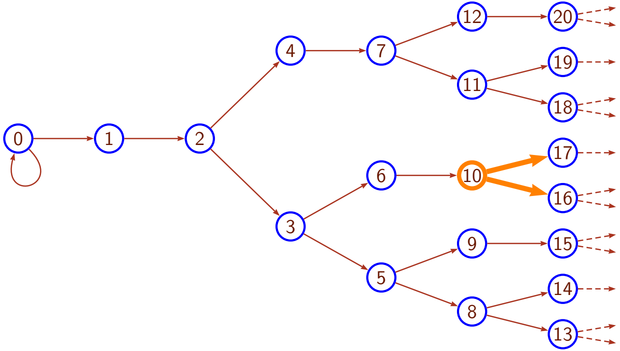

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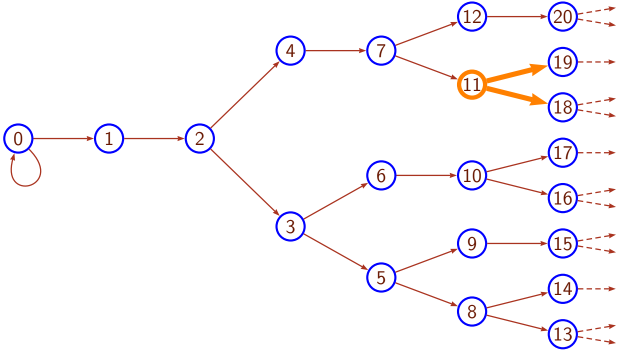

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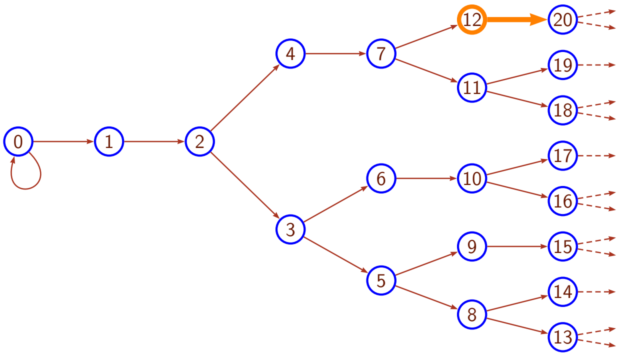
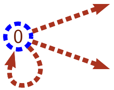


$$\mathbf{s} = 2\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ \cdots$$

$$\mathbf{s} = (\ 3\ \ 2\ \ 1\ )^{\omega}$$



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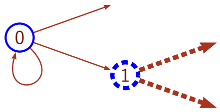

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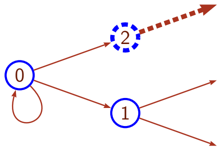

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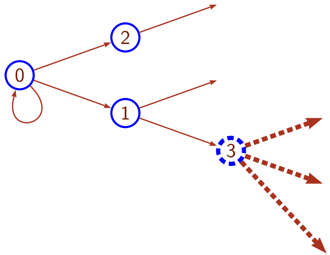

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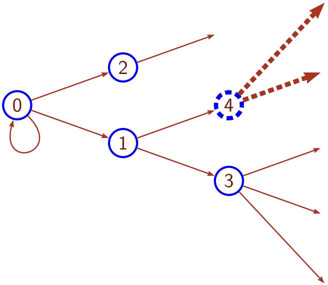

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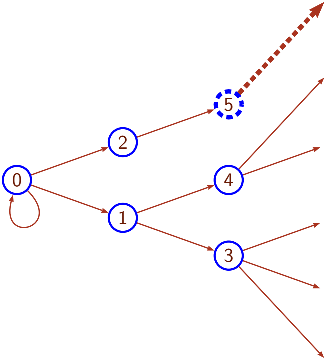

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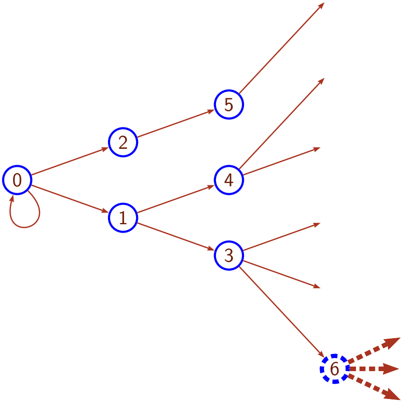

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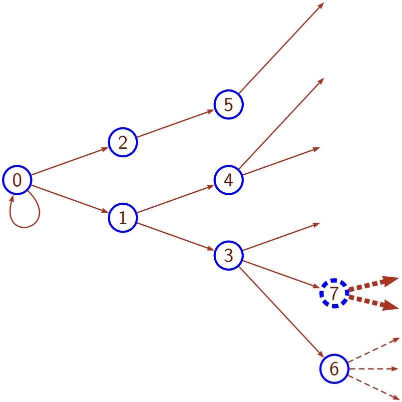

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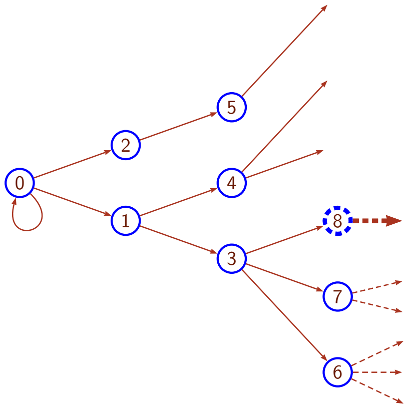

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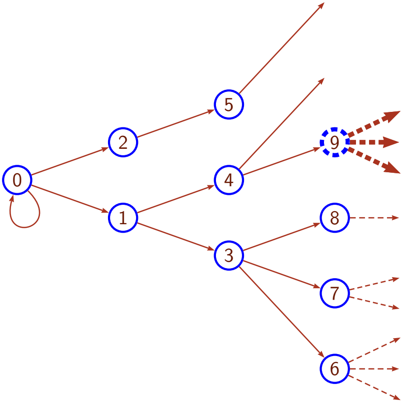

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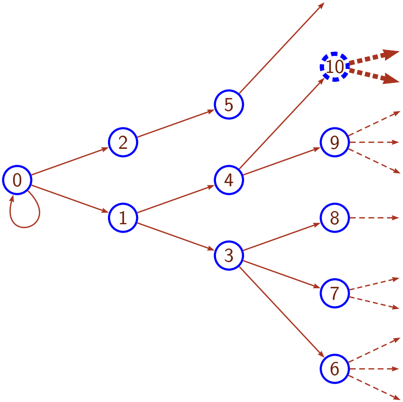

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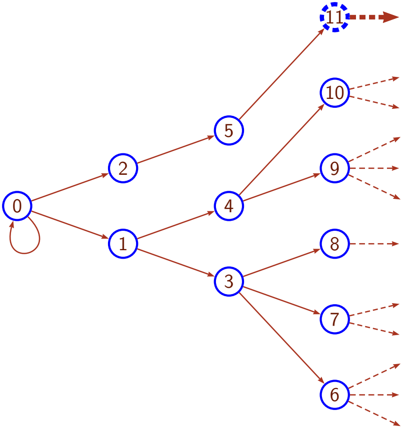

Alphabets are ordered hence

prefix-closed languages $=$ labelled trees.

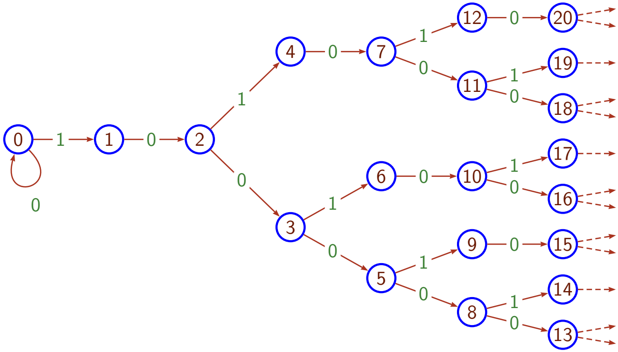

Figure : Integer representations in the Fibonacci numeration system.

Prefix-closed languages and labelled trees

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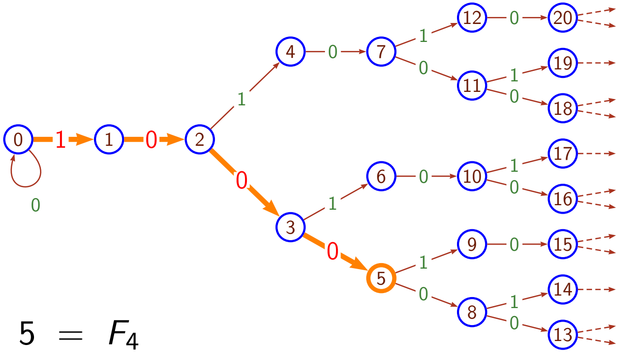

$5 = F_4$

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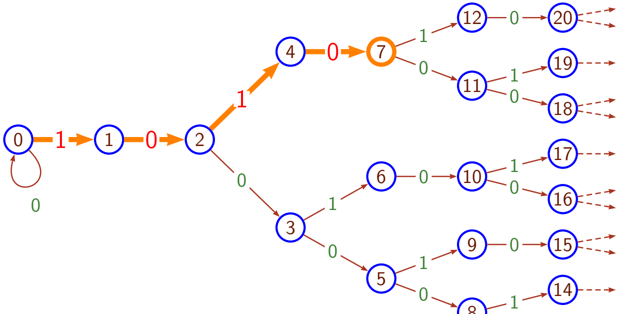

$$7 = 5 + 2 = F_4 + F_2$$

Figure : Integer representations in the Fibonacci numeration system.

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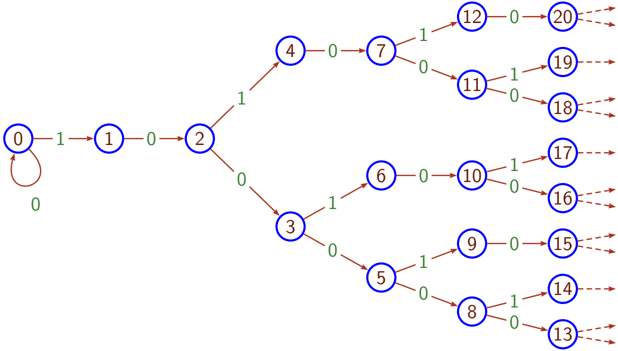

$\mathbf{s} =$

$\lambda =$

Serialisation of a prefix-closed language

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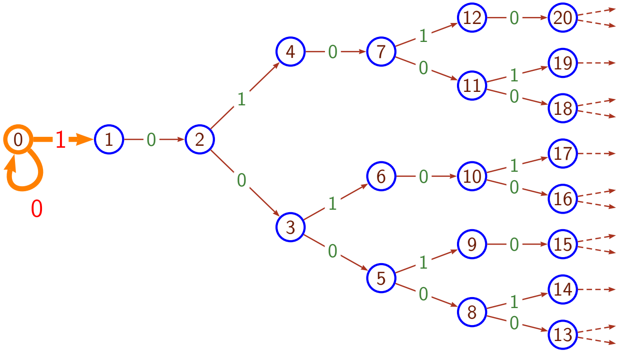

$\mathbf{s} = 2$

$\lambda = 01$

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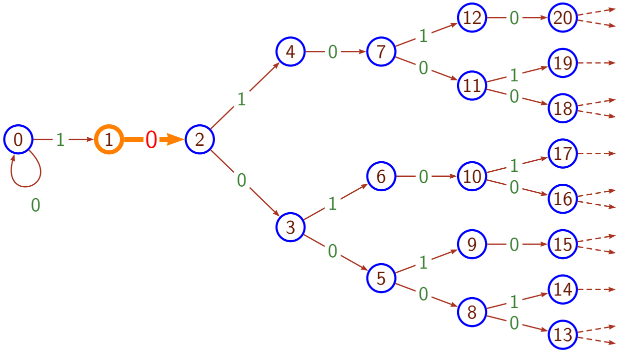

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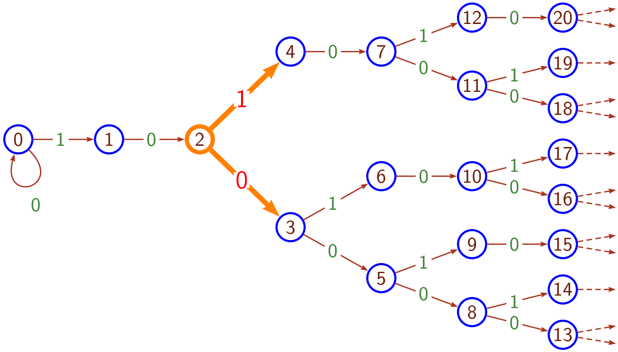

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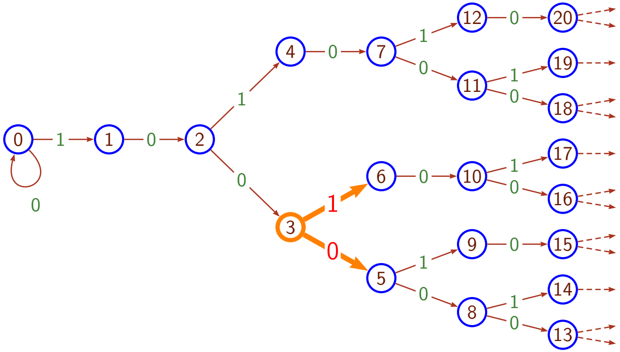

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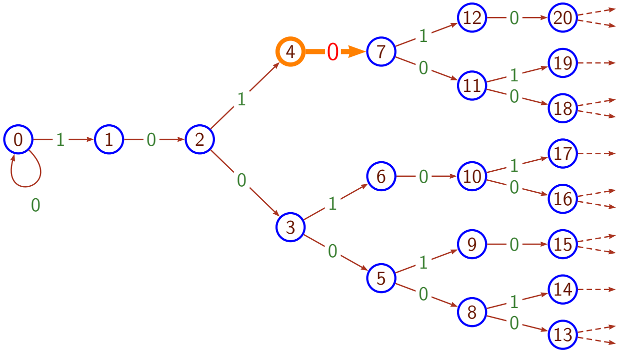

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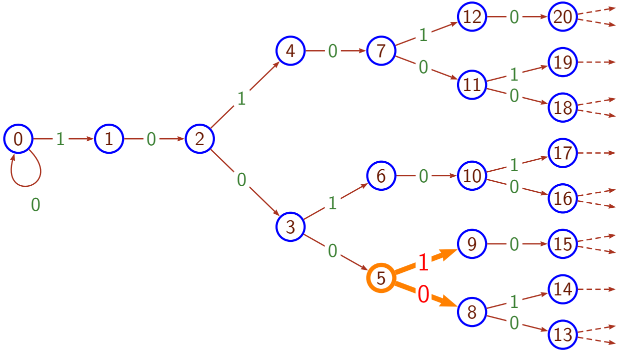

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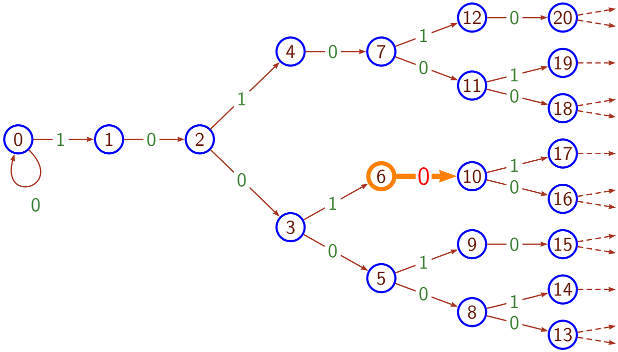

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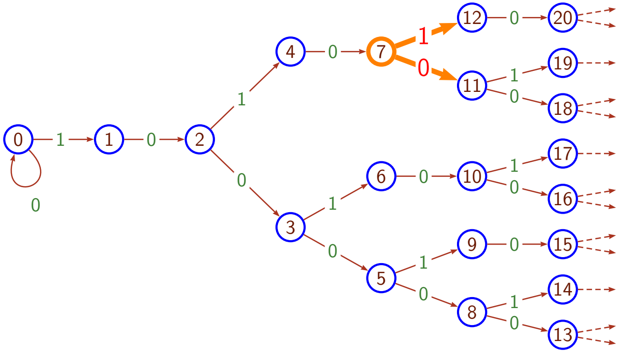

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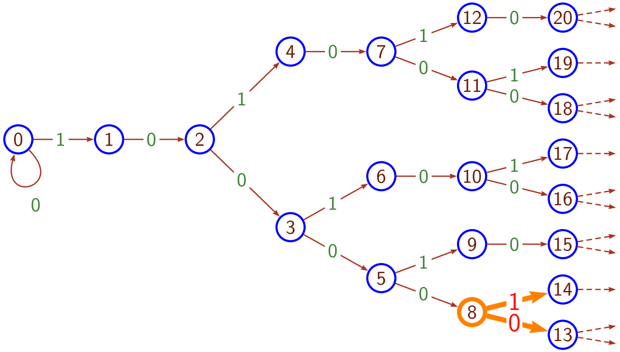

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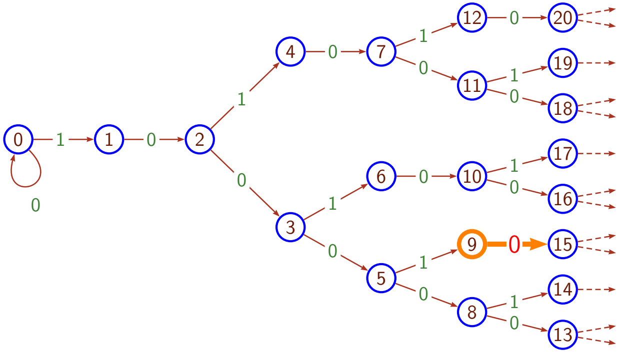

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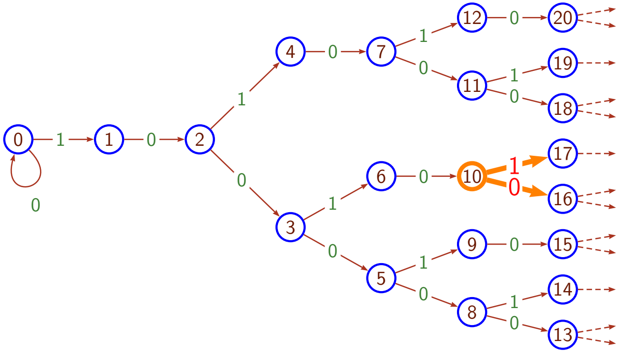

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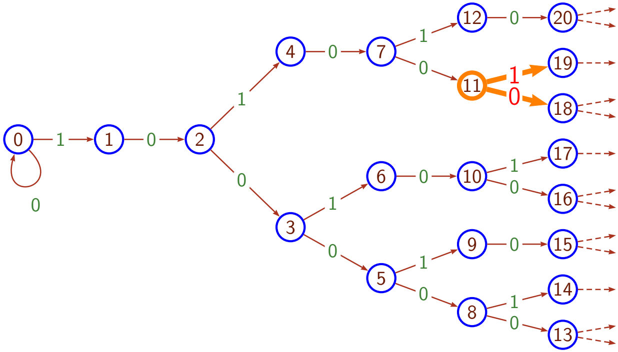

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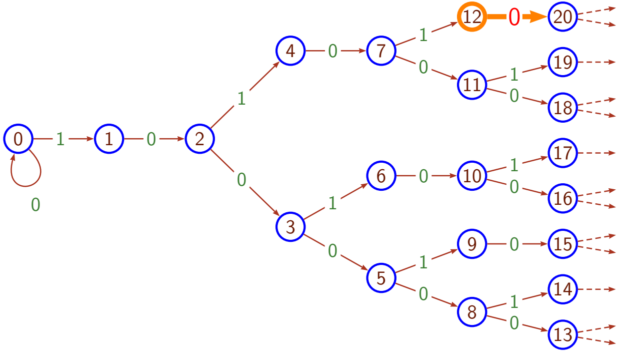
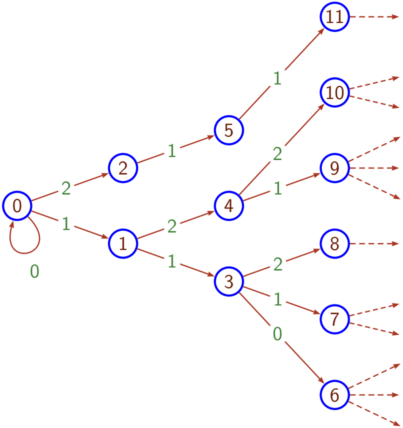


$$\mathbf{s} = 2\ 1\ 2\ \ 2\ 1\ 2\ 1\ 2\ \ 2\ 1\ 2\ \ 2\ 1\ \cdots$$

$$\lambda = 01\ 0\ 01\ 01\ 0\ 01\ 0\ 01\ 01\ 0\ 01\ 01\ 0\ \cdots$$

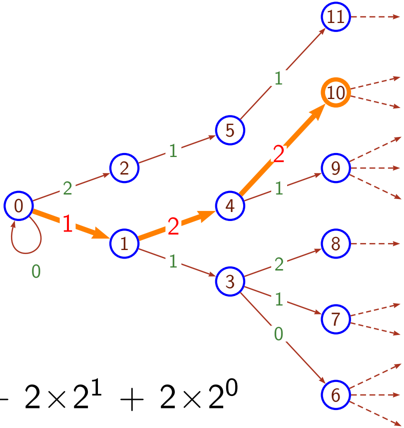
The pair signature/labelling is characteristic

$$\mathbf{s} = (3\ 2\ 1)^\omega$$
$$\lambda = (012\ 12\ 1)^\omega$$



The pair signature/labelling is characteristic

$$\mathbf{s} = (3\ 2\ 1)^\omega$$
$$\lambda = (012\ 12\ 1)^\omega$$



$$10 = 1\times 2^2 + 2\times 2^1 + 2\times 2^0$$

Figure : Non-canonical integer representations in base 2.

Theorem

L: a prefix-closed language.
Signature(L) is substitutive $\Leftrightarrow$ L is accepted by a finite automaton.

A word on substitution

A substitution σ is a morphism $A^* \to A^*$.

Running examples

Fibonacci substitution: $\{a, b\} \to \{a, b\}^*$
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In this case, $\sigma^\omega(a)$ exists and is called a purely substitutive word.

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let f_σ be the (letter-to-letter) morphism: $A^* \to \mathbb{N}^*$ defined by

- $\forall b,\ f_\sigma(b) = |\sigma(b)|$

We call $f_\sigma(\sigma^\omega(a))$ a **subtitutive signature**.

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If g is a morphism such that

- $\forall b,\ |g(b)| = |\sigma(b)|$
- if $g(b) = c_0 c_1 \cdots c_k$ then $c_0 < c_1 < \cdots < c_k$

We call $g(\sigma^\omega(a))$ a **substitutive labelling**.

Example 1 – the Fibonacci signature

$$\sigma(a) = ab \implies f_\sigma(a) = 2$$
$$\sigma(b) = a \implies f_\sigma(b) = 1$$
$$f_\sigma(\sigma^\omega(a)) = 2122121221221212212122\cdots$$

if we choose g:

$g(a) = 01$

$g(b) = 0$

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This pair signature/labelling defines the language of integer representations in the Fibonacci numeration system.

Example 2 – a periodic signature

$\sigma(a) = abc \quad (f_\sigma(a) = 3)$
$\sigma(b) = ab \quad (f_\sigma(b) = 2)$
$\sigma(c) = c \quad (f_\sigma(c) = 1)$

$$\sigma(abc) = abc\,abc \qquad \text{hence } f_\sigma(\sigma^\omega(a)) = (321)^\omega$$

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This pair signature/labelling defines a *non-canonical* representation of integers in base 2.

Example 3 – the Thue-Morse morphism

$$\sigma(a) = ab \quad (f_\sigma(a) = 2)$$
$$\sigma(b) = ba \quad (f_\sigma(b) = 2)$$
$$f_\sigma(\sigma^\omega(a)) \quad = \quad 2^\omega$$

$\forall$ labelling g, the language is *essentially* $(0+1)^*$.

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Theorem

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Automaton associated with a subst. signature

$\sigma : A^* \to A^*$ prolongable on a and $g : A^* \to B^*$

$$\mathcal{A}_{(\sigma,g)} = \langle \mathsf{A}, B, \delta, \{\mathsf{a}\}, \mathsf{A} \rangle$$

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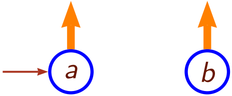

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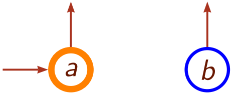

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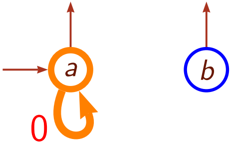

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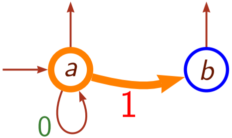

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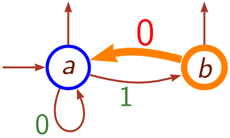

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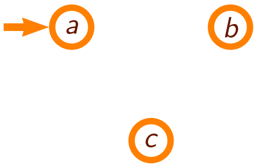

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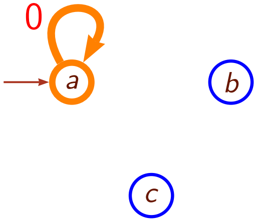

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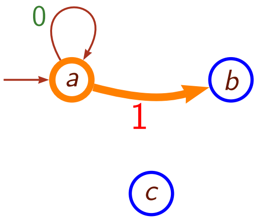

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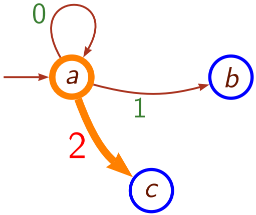

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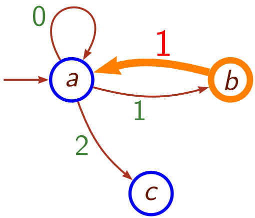

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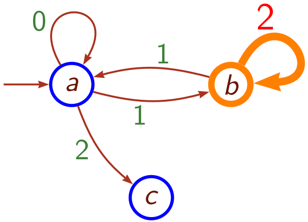

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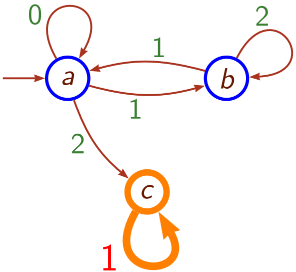

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Proof: unfold the automaton $\mathcal{A}_{(\sigma,g)}$.

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Abstract Numeration System:
built from an arbitrary regular language.

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Theorem (augmented version)

Every DTNS is a prefix-closed ANS.

Every prefix-closed ARNS is easily† convertible to a DTNS.

† Through a finite, letter-to-letter and pure sequential transducer.

Other works: Ultimately periodic signatures

$$\mathbf{s} \;=\; u\, r^{\omega} \qquad \text{with} \qquad r \;=\; r_0\, r_1\, r_2\, \cdots\, r_{q-1}$$

Definition: growth ratio

$$\mathtt{gr}(\mathbf{s}) \quad = \quad \frac{r_0 + r_1 + \cdots + r_{q-1}}{q}$$

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Theorem (MS, to appear)

If $\mathtt{gr}(\mathbf{s}) \in \mathbb{N}$, then **s** generates the language of a finite automaton.
It is linked‡ to the integer base $b = \mathtt{gr}(\mathbf{s})$.

If $\mathtt{gr}(\mathbf{s}) \notin \mathbb{N}$, then **s** generates a non-context-free language.
It is linked‡ to the *rational base* $\frac{p}{q} = \mathtt{gr}(\mathbf{s})$. (cf. Akiyama et al. '08)

‡ It is a non-canonical representation of the integers (using extra digits).

Future works : Directed signatures

Aperiodic signature: $\mathbf{s} = s_0\, s_1\, s_2 \cdots$

$S_n = \frac{1}{n}\Sigma_{k=0}^{n-1} s_k$: partial average of **s**.
α : lim S_n extends the notion of growth ratio.