On automatic transitive graphs

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Introduction

- Finite automata presentable finitely generated groups
- Automatic groups
- Cayley automatic groups
- Automatic transitive graphs

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Finite automata presentable groups

Definition

Let G be a group. We say that G is finite automata presentable if there exist a finite state automaton W over some alphabet Σ and a surjective map $\tau : L(W) \to G$ such that the equality relation $\tau(w_1) = \tau(w_2)$ and the group operation $\tau(w_1)\tau(w_2) = \tau(w_3)$ are finite automata presentable.

Theorem (Oliver, Thomas, 2005)

A finitely generated group G is FA–presentable if and only if it is virtually abelian.

Automatic groups

Definition

Let G be a finitely generated group. An automatic structure on G consists of a set A of semigroup generators of G, a finite state automaton W over A, and finite state automata M_x over (A, A), for $x \in A \cup \{e\}$, satisfying the following conditions:

- The map $\pi : L(W) \to G$ is surjective.
- For $x \in A \cup \{e\}$, we have $(w_1, w_2) \in L(M_x)$ if and only if $\pi(w_1)x = \pi(w_2)$ and both w_1 and w_2 are elements of $L(W)$.

We say that G is automatic (in the sense of Thurston) if it admits an automatic structure.

D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston
Word Processing in Groups, Jones and Barlett Publishers. Boston, MA, 1992.

Cayley automatic groups

Definition

Let G be a group generated by a finite set X. We say that G is Cayley automatic if the directed and labeled Cayley graph $\Gamma(G, X)$ is finite automata presentable.

O. Kharlampovich, B. Khoussainov, A. Miasnikov
From automatic structures to automatic groups,
arXiv:1107.3645v2 [math.GR], 2011.

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Examples of Cayley automatic groups that are not automatic

The Heisenberg group

The three–dimensional Heisenberg group:

$$H_3 = \{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{Z} \}.$$

The Baumslag–Solitar group

The Baumslag–Solitar group $BS(1,2) = \langle a, t \mid t^{-1}at = a^2 \rangle$.

The lamplighter group

The lamplighter group is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$.

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Automatic transitive graphs

Definition

Let G be a connected locally finite transitive graph. We say that G is automatic if it is finite automata presentable.

Remark

Removing labels from an undirected Cayley graph of a Cayley automatic group we obtain an automatic transitive graph.

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Summary

- Baumslag–Solitar groups
- Wreath products
- Some examples of automatic transitive non–Cayley graphs. The Deistel–Leader graph.

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The Baumslag–Solitar groups

Observation

The Baumslag–Solitar group $BS(m, n)$ is the HNN extension of $\mathbb{Z} = \langle a \rangle$ relative to subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$, and the isomorphism $\varphi : m\mathbb{Z} \to n\mathbb{Z}$ that maps a^m to a^n.

Proposition

Every element $w \in BS(m, n)$ has a unique representation as

$$w = g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} g_0,$$

where $g_0 = a^k$ for some $k \in \mathbb{Z}$, and if $\epsilon_i = -1$ then $g_i \in \{1, a, \dots, a^{n-1}\}$, if $\epsilon_i = +1$ then $g_i \in \{1, a, \dots, a^{m-1}\}$, and there is no consecutive subsequence $t^\epsilon, 1, t^{-\epsilon}$.

Theorem

The group $BS(m, n)$ is Cayley automatic.

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The idea of the proof

The right–multiplication by the generator a

The right–multiplication by the generator a transforms the normal form of w as follows: $g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^k \xrightarrow{\times a} g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^{k+1}$.

The right–multiplication by the generator t

Let $k = mp + r$ where $p \in \mathbb{Z}$ and $r \in \{0, \dots, m-1\}$. The right–multiplication by the generator t transforms a normal form of w as follows

- if $r \neq 0$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^r t a^{np},$$

- if $r = 0$, and $\ell \geqslant 1, \epsilon_1 = -1$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_2 t^{\epsilon_2} g_1 t^{-1} a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_2 t^{\epsilon_2} (g_1 a^{np}),$$

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The idea of the proof (continue)

The right–multiplication by the generator t

- if $r = 0$ and $\ell \geqslant 1, \epsilon_1 = 1$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_1 t \, 1 \, t a^{np},$$

- if $r = 0$ and $\ell = 0$ then

$$a^k \xrightarrow{\times t} a^r t a^{np}.$$

Remark

We use the m–ary representation of $k \in \mathbb{Z}$ for which the map $k = mp + r \to np$ is finite automata presentable.

Remark

If $m = n$, the map $k = mp + r \to mp$ is finite automata presentable for the unary representation of k; therefore, $BS(m, n)$ is automatic.

The wreath products $G \wr \mathbb{Z}$

Multiplication in a group $G \wr \mathbb{Z}$

Represent each element of $G \wr \mathbb{Z}$ as a pair (f, z), where $f \in G^{(\mathbb{Z})}$ and $z \in \mathbb{Z}$. The multiplication in $G \wr \mathbb{Z}$ is given by: $(f, z)(f', z') = (f\, f'^{-z}, z + z')$, where $f'^{-z}(x) = f'(x - z)$.

Representation of elements of $G \wr \mathbb{Z}$

We represent an element of $G \wr \mathbb{Z}$ as a finite string of the form:

$$v_{-j}\# \dots \# v_{-1} A v_0 \# v_1 \# \dots \# v_{m-1} C v_m \# v_{m+1} \dots \# v_i,$$

where $v_{-j}, \dots, v_i$ are the words of a regular language P that gives a Cayley automatic representation of G, and v_{-j} and v_i are the representatives for the leftmost and rightmost nontrivial elements of G.

The wreath products $G \wr \mathbb{Z}$ (continue)

The right–multiplication by $(\mathbf{e}, t)$ corresponds to the following relation:

$$\left(\begin{array}{ccccccccc} \dots & \# & v_{m-1} & C & v_m & \# & v_{m+1} & \# & \dots \\ \dots & \# & v_{m-1} & \# & v_m & C & v_{m+1} & \# & \dots \end{array} \right).$$

The right–multiplication by $(f_i, 0)$ corresponds to the following relation:

$$\left(\begin{array}{ccccccccc} \dots & \# & v_{m-1} & C & v_m & \# & v_{m+1} & \# & \dots \\ \dots & \# & v_{m-1} & C & u_m & \# & v_{m+1} & \# & \dots \end{array} \right).$$

Theorem

For a Cayley automatic group G the wreath product $G \wr \mathbb{Z}$ is Cayley automatic.

Is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}^2$ Cayley automatic?

What would it mean for $\mathbb{Z}_2 \wr \mathbb{Z}^2$ to be Cayley automatic?

Suppose that P is a regular language uniquely representing elements of $\mathbb{Z}_2 \wr \mathbb{Z}^2$. Let $f_0 \in \mathbb{Z}_2^{(\mathbb{Z}^2)}$ be the function such that $f_0(0,0) = 1$ and $f_0(z_1, z_2) = 0$ if $(z_1, z_2) \neq (0,0)$; let $r = (1,0) \in \mathbb{Z}^2$ and $u = (0,1) \in \mathbb{Z}^2$. These are the generators of $\mathbb{Z}_2 \wr \mathbb{Z}^2$. Cayley automaticity of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ implies that the right–multiplications by f_0, r and u are finite automata presenatable.

Proposition

Suppose that $\mathbb{Z}_2 \wr \mathbb{Z}^2$ is Cayley automatic with respect to P such that the subset $P' \subset P$ of representatives of the subgroup $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ is a regular language. Then the group operation in $\mathbb{Z}_2^{(\mathbb{Z}^2)}$ is not finite automata presentable.

The example of automatic transitive graph which is not a Cayley graph

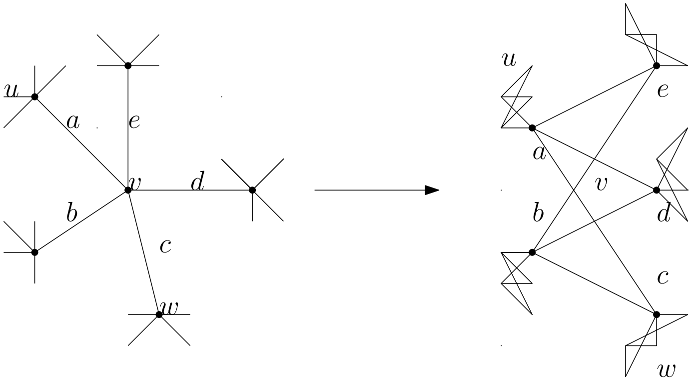

Figure: Constructing the non–Cayley graph $H_{2,3}$ from T_5 (Thomassen, C., Watkins, M.E., 1989)

The Diestel–Leader graph G

Description of the Diestel–Leader graph G

Let X be a 3–regular tree in which each node has in–degree 2 and out–degree 1. Let Y be a 4–regular tree in which each node has in–degree 1 and out–degree 3. Fix a vertex $O_1 \in V(X)$ and a vertex $O_2 \in V(Y)$. For each $x \in X$, set $r(x)$ to be the signed distance from O_1 to x, i.e., if the unique undirected path from O_1 to x in X has s forward edges and t backward edges then $r(x) = s - t$. Define $r(y)$ similarly for each $y \in Y$. The set of vertices of G is the set $\{(x, y) \in X \times Y : r(x) = r(y)\}$, and G has an arc from (x, y) to (x', y') if $(x, x') \in E(X)$ and $(y, y') \in E(Y)$.

Theorem

The Diestel–Leader graph G is automatic.

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Acknowledgements

Thank You!

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