Closure Properties of Pattern Languages

Daniel Reidenbach[1], Joel D. Day[1], **Markus L. Schmid**[2]

[1]Loughborough University, UK
[2]Trier University, Germany

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Basic Definitions and Notation

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Σ	*Terminals*	$\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$
X	*Variables*	$\{x_1, x_2, x_3, \ldots\}$
$w \in \Sigma^*$	*Word*	$\mathtt{abaacba}$
$\alpha \in (\Sigma \cup X)^+$	*Pattern*	$\alpha := x_1 \mathtt{a} x_2 x_1 \mathtt{ba} x_2 x_1 x_3$

Pattern Languages

Morphism: Mapping $h : \Gamma_1^* \to \Gamma_2^*$ with $h(x \cdot y) = h(x) \cdot h(y)$;
h is nonerasing iff, for every $a \in \Gamma_1$, $h(a) \neq \varepsilon$.

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E-pattern lang.	$L_{\mathsf{E},\Sigma}(\alpha) := \{h(\alpha) \mid h$ is a substitution$\}$.
NE-pattern lang.	$L_{\mathsf{NE},\Sigma}(\alpha) := \{h(\alpha) \mid h$ is nonerasing substitution$\}$.

An Example

$\alpha = x_1 \; \texttt{aa} \; x_2 \; x_1 \; x_2 \; \texttt{cb} \; x_1$

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$h(\alpha) = \mathtt{acaaabcbaacabcbacbac} \in L_{\mathsf{NE},\{\mathtt{a},\mathtt{b},\mathtt{c}\}}(\alpha)$,
where $h(x_1) = \mathtt{ac}$, $h(x_2) = \mathtt{abcba}$, $(h(\mathtt{a}) = \mathtt{a}$, $h(\mathtt{b}) = \mathtt{b})$.

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- Later investigated from a purely language theoretical point of view.
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- Practically applied in so-called regular expressions with backreferences (Perl, Java, Python, ...).
- Relations to combinatorics on words: pattern avoidability, ambiguity of morphisms, word equations, equality sets.

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Closure Properties

Angluin 1979:

Pattern Languages are not closed under

- union $L_{\mathrm{NE},\Sigma}(\mathtt{a}) \cup L_{\mathrm{NE},\Sigma}(\mathtt{b}) = \{\mathtt{a}, \mathtt{b}\}$
- intersection $L_{\mathrm{NE},\Sigma}(\mathtt{a}) \cap L_{\mathrm{NE},\Sigma}(\mathtt{b}) = \emptyset$
- complement $\{\mathtt{a}, \mathtt{b}\}^* \setminus L_{\mathrm{NE},\Sigma}(\mathtt{a})$
- Kleene plus $(L_{\mathrm{NE},\{\mathtt{a},\mathtt{b}\}}(\mathtt{a}))^*$ $((L_{\mathrm{NE},\{\mathtt{a},\mathtt{b}\}}(\mathtt{a}))^+)$
- homomorphism $h(L_{\mathrm{NE},\{\mathtt{a},\mathtt{b}\}}(x)) = (L(\mathtt{a}))^+$, $h(\mathtt{a}) = h(\mathtt{b}) = \mathtt{a}$
- inv. homo. $g^{-1}(L_{\mathrm{NE},\{\mathtt{a},\mathtt{b}\}}(\mathtt{aaa})) = \{\mathtt{aaa}, \mathtt{ab}, \mathtt{ba}\}$, $g(\mathtt{a}) = \mathtt{a}$, $g(\mathtt{b})$

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Pattern Languages are closed under

- concatenation $L(\alpha) \cdot L(\beta) = L(\alpha \cdot \beta)$
- reversal $(L(\alpha))^R = L(\alpha^R)$

Motivation for Investigating Closure Properties

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- In the case of pattern languages the existing closure properties fail to contribute to our understanding of their intrinsic properties.
- All examples for non-closure require terminal symbols in the patterns (what about the closure of terminal-free pattern languages).
- Can we characterise those pairs (α, β) of patterns, for which $L(\alpha) \cup L(\beta)$ or $L(\alpha) \cap L(\beta)$ are pattern languages?

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$$\begin{aligned}
\beta_1 &= x_1 x_2 x_2 x_1 x_3 x_1, \\
\beta_2 &= x_2 x_2 x_3, \\
\beta_3 &= x_1 x_1 x_3 x_1, \\
\beta_4 &= x_1 x_2 x_2 x_1 x_1, \\
\beta_5 &= x_3, \\
\beta_6 &= x_2 x_2, \\
\beta_7 &= x_1 x_1 x_1.
\end{aligned}$$

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$$L_{\mathrm{E},\Sigma}(\alpha) = \bigcup_{i=1}^{6} L_{\mathrm{NE},\Sigma}(\beta_i).$$

$$\begin{aligned}
\gamma_1 &= a x_1\, a x_2\, a x_2\, a x_1\, a x_3\, a x_1, \\
\gamma_2 &= b x_1\, a x_2\, a x_2\, b x_1\, a x_3\, b x_1, \\
\gamma_3 &= a x_1\, b x_2\, b x_2\, a x_1\, a x_3\, a x_1, \\
\gamma_4 &= a x_1\, a x_2\, a x_2\, a x_1\, b x_3\, a x_1, \\
\gamma_5 &= a x_1\, b x_2\, b x_2\, a x_1\, b x_3\, a x_1, \\
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\gamma_7 &= b x_1\, b x_2\, b x_2\, b x_1\, a x_3\, b x_1, \\
\gamma_8 &= b x_1\, b x_2\, b x_2\, b x_1\, b x_3\, b x_1.
\end{aligned}$$

$$L_{\mathrm{NE},\Sigma}(\alpha) = \bigcup_{i=1}^{8} L_{\mathrm{E},\Sigma}(\gamma_i).$$

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\gamma_1 &= ax_1\, ax_2\, ax_2\, ax_1\, ax_3\, ax_1, \\
\gamma_2 &= bx_1\, ax_2\, ax_2\, bx_1\, ax_3\, bx_1, \\
\gamma_3 &= ax_1\, bx_2\, bx_2\, ax_1\, ax_3\, ax_1, \\
\gamma_4 &= ax_1\, ax_2\, ax_2\, ax_1\, bx_3\, ax_1, \\
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$$L_{\mathrm{NE},\Sigma}(\alpha) = \bigcup_{i=1}^{8} L_{\mathrm{E},\Sigma}(\gamma_i).$$

Is this the only way of how unions of E- or unions of NE- pattern languages can be a NE- or a E-pattern languages, respectively?

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- ... have open closure properties.

Union of Terminal-Free Pattern Languages

Theorem

Let Z, $Z' \in \{\mathrm{E}, \mathrm{NE}\}$ *and* α, β, γ *patterns.*

$$L_{Z,\Sigma}(\alpha) \cup L_{Z,\Sigma}(\beta) = L_{Z',\Sigma}(\gamma)$$

$$\Longleftrightarrow$$

$$L_{Z,\Sigma}(\alpha) \subseteq L_{Z,\Sigma}(\beta) \text{ and } L_{Z,\Sigma}(\beta) = L_{Z',\Sigma}(\gamma) \text{ or}$$
$$L_{Z,\Sigma}(\beta) \subseteq L_{Z,\Sigma}(\alpha) \text{ and } L_{Z,\Sigma}(\alpha) = L_{Z',\Sigma}(\gamma)\,.$$

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$\Rightarrow$ full characterisation of $L_Z(\alpha) \cup L_Z(\beta) = L_{Z'}(\gamma)$, $Z, Z' \in \{\mathsf{E}, \mathsf{NE}\}$.

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Inclusion is decidable for terminal-free E-pattern languages, but still open for terminal-free NE-pattern languages

Intersection of Terminal-Free Pattern Languages

Theorem

Let $Z \in \{\mathsf{E}, \mathsf{NE}\}$*. Then* $L_{Z,\Sigma}(x_1 x_1) \cap L_{Z,\Sigma}(x_1 x_1 x_1) = L_{Z,\Sigma}(x_1^6)$.

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$L_{\mathsf{NE},\Sigma}(x_1 x_2 x_1) \cap L_{\mathsf{NE},\Sigma}(x_1 x_1 x_2)$ *is not a terminal-free* NE*-pattern language.*

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Theorem

$L_{\mathrm{E},\Sigma}(x_1 x_2 x_1^2 x_2 x_1^3 x_2^2) \cap L_{\mathrm{E},\Sigma}(x_3 x_4^2 x_3^2 x_4^6 x_3^3)$ *is not a tf-*E*-pattern language.*

Proof Sketch

Let $\alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2$ and $\beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3$.

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Let $\alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2$ and $\beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3$.

$L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ equals the solutions of

$$x_1 x_2 x_1 x_1 x_2 x_1 x_1 x_1 x_2 x_2 = x_3 x_4 x_4 x_3 x_3 x_4 x_4 x_4 x_4 x_4 x_4 x_3 x_3 x_3 \; .$$

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$L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ equals the solutions of

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$$x_5 = x_4 x_4$$

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$\Rightarrow$ all solutions to the equations are periodic.

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Lemma: If $\alpha = \beta$ has only periodic solutions and $L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ is a terminal-free E-pattern language, then $a^k \in L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ implies $k = \ell |w|$ for some $\ell \geq 1$.

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Since a^6 is the shortest element in $L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ and $\mathsf{a}^8 \in L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$, we obtain a contradiction.

Other Closure Properties of TF Pattern Languages

Theorem

Let $|\Sigma| \geq 2$. *The terminal-free* NE- *and* E-*pattern languages, with respect to* Σ, *are not closed under*

- *morphisms,*
- *inverse morphisms,*
- *Kleene plus and*
- *Kleene star.*

Theorem

For every terminal-free pattern α, *the complement of* $L_{\mathsf{E},\Sigma}(\alpha)$ *is not a terminal-free* E-*pattern language and the complement of* $L_{\mathsf{NE},\Sigma}(\alpha)$ *is not a terminal-free* NE-*pattern language.*

Closure Properties of General Pattern Languages

Closure under complement is fully characterised:

Theorem

For every pattern α, the complement of $L_{\mathsf{E},\Sigma}(\alpha)$ is not an E-pattern language and the complement of $L_{\mathsf{NE},\Sigma}(\alpha)$ is not a NE-pattern language.

Main Research Question

For $Z, Z' \in \{\text{E}, \text{NE}\}$ and $\circ \in \{\cup, \cap\}$, are there α, β such that

- $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is not a Z'-pattern language? ✓
- $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is a Z'-pattern language?

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For $Z, Z' \in \{\mathrm{E}, \mathrm{NE}\}$ and $\circ \in \{\cup, \cap\}$, are there α, β such that

- $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is not a Z'-pattern language? ✓
- $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is a Z'-pattern language?

Characterise the α, β for which $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is a Z'-pattern language?

Intersection of General Pattern Languages

There are simple examples for the situation that

- $L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ is an E-pattern language.
- $L_{\mathsf{NE},\Sigma}(\alpha) \cap L_{\mathsf{NE},\Sigma}(\beta)$ is an NE-pattern language.
- $L_{\mathsf{NE},\Sigma}(\alpha) \cap L_{\mathsf{NE},\Sigma}(\beta)$ is an E-pattern language.

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Open:

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Open:

- Are there α, β, such that $L_{\mathsf{E},\Sigma}(\alpha) \cap L_{\mathsf{E},\Sigma}(\beta)$ is NE-pattern language?
- Characterisations?

Union of General Pattern Languages

There are simple examples for the situation that

- $L_{\mathsf{NE},\Sigma}(\alpha) \cup L_{\mathsf{NE},\Sigma}(\beta)$ is an NE-pattern language.
- $L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta)$ is an NE-pattern language.
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Union of General Pattern Languages

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- $L_{\mathrm{NE},\Sigma}(\alpha) \cup L_{\mathrm{NE},\Sigma}(\beta)$ is an NE-pattern language.
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- $L_{\mathrm{NE},\Sigma}(\alpha) \cup L_{\mathrm{NE},\Sigma}(\beta)$ is an E-pattern language.

Examples for the situation that $L_{\mathrm{E},\Sigma}(\alpha) \cup L_{\mathrm{E},\Sigma}(\beta)$ is an E-pattern language exist, but are much more complicated.

Union of General Pattern Languages

Example for "$\mathsf{E} \cup \mathsf{E} = \mathsf{E}$" and alphabet size 2:

$$\Sigma = \{\mathtt{a}, \mathtt{b}\},$$
$$\alpha = x_1 \mathtt{a} x_2 \mathtt{b} x_2 \mathtt{a} x_3,$$
$$\beta = x_1 \mathtt{a} x_2 \mathtt{b}\mathtt{b} x_2 \mathtt{a} x_3,$$
$$\gamma = x_1 \mathtt{a} x_2 \mathtt{b} x_3 \mathtt{a} x_4.$$

$$L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma),$$
$$L_{\mathsf{E},\Sigma}(\alpha) \not\subseteq L_{\mathsf{E},\Sigma}(\beta),$$
$$L_{\mathsf{E},\Sigma}(\beta) \not\subseteq L_{\mathsf{E},\Sigma}(\alpha).$$

Union of General Pattern Languages

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$\Sigma = \{\mathtt{a}, \mathtt{b}\}$,
$\alpha = x_1 \mathtt{a} x_2 \mathtt{b} x_2 \mathtt{a} x_3$,
$\beta = x_1 \mathtt{a} x_2 \mathtt{bb} x_2 \mathtt{a} x_3$,
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$L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma)$,
$L_{\mathsf{E},\Sigma}(\alpha) \not\subseteq L_{\mathsf{E},\Sigma}(\beta)$,
$L_{\mathsf{E},\Sigma}(\beta) \not\subseteq L_{\mathsf{E},\Sigma}(\alpha)$.

Proof sketch:

Union of General Pattern Languages

Example for "$E \cup E = E$" and alphabet size 2:

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$L_{E,\Sigma}(\alpha) \cup L_{E,\Sigma}(\beta) = L_{E,\Sigma}(\gamma)$,
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Proof sketch:

$L_{E,\Sigma}(\alpha) \subseteq L_{E,\Sigma}(\gamma)$ and
$L_{E,\Sigma}(\beta) \subseteq L_{E,\Sigma}(\gamma)$ is obvious.

Union of General Pattern Languages

Example for "E ∪ E = E" and alphabet size 2:

$\Sigma = \{\mathtt{a}, \mathtt{b}\}$,
$\alpha = x_1 \mathtt{a} x_2 \mathtt{b} x_2 \mathtt{a} x_3$,
$\beta = x_1 \mathtt{a} x_2 \mathtt{bb} x_2 \mathtt{a} x_3$,
$\gamma = x_1 \mathtt{a} x_2 \mathtt{b} x_3 \mathtt{a} x_4$.

$L_{\mathrm{E},\Sigma}(\alpha) \cup L_{\mathrm{E},\Sigma}(\beta) = L_{\mathrm{E},\Sigma}(\gamma)$,
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Let $w \in L_{\mathrm{E},\Sigma}(\gamma)$

Union of General Pattern Languages

Example for "E ∪ E = E" and alphabet size 2:

$\Sigma = \{\mathtt{a}, \mathtt{b}\}$,
$\alpha = x_1 \mathtt{a} x_2 \mathtt{b} x_2 \mathtt{a} x_3$,
$\beta = x_1 \mathtt{a} x_2 \mathtt{b}\mathtt{b} x_2 \mathtt{a} x_3$,
$\gamma = x_1 \mathtt{a} x_2 \mathtt{b} x_3 \mathtt{a} x_4$.

$L_{\mathrm{E},\Sigma}(\alpha) \cup L_{\mathrm{E},\Sigma}(\beta) = L_{\mathrm{E},\Sigma}(\gamma)$,
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Proof sketch:

$L_{\mathrm{E},\Sigma}(\alpha) \subseteq L_{\mathrm{E},\Sigma}(\gamma)$ and
$L_{\mathrm{E},\Sigma}(\beta) \subseteq L_{\mathrm{E},\Sigma}(\gamma)$ is obvious.

Let $w \in L_{\mathrm{E},\Sigma}(\gamma)$
$w = u\, \mathtt{a}\, \mathtt{b}^n\, \mathtt{a}\, v$,

Union of General Pattern Languages

Example for "$\mathrm{E} \cup \mathrm{E} = \mathrm{E}$" and alphabet size 2:

$\Sigma = \{\mathtt{a}, \mathtt{b}\}$,
$\alpha = x_1 \mathtt{a} x_2 \mathtt{b} x_2 \mathtt{a} x_3$,
$\beta = x_1 \mathtt{a} x_2 \mathtt{bb} x_2 \mathtt{a} x_3$,
$\gamma = x_1 \mathtt{a} x_2 \mathtt{b} x_3 \mathtt{a} x_4$.

$L_{\mathrm{E},\Sigma}(\alpha) \cup L_{\mathrm{E},\Sigma}(\beta) = L_{\mathrm{E},\Sigma}(\gamma)$,
$L_{\mathrm{E},\Sigma}(\alpha) \not\subseteq L_{\mathrm{E},\Sigma}(\beta)$,
$L_{\mathrm{E},\Sigma}(\beta) \not\subseteq L_{\mathrm{E},\Sigma}(\alpha)$.

Proof sketch:

$L_{\mathrm{E},\Sigma}(\alpha) \subseteq L_{\mathrm{E},\Sigma}(\gamma)$ and
$L_{\mathrm{E},\Sigma}(\beta) \subseteq L_{\mathrm{E},\Sigma}(\gamma)$ is obvious.

Let $w \in L_{\mathrm{E},\Sigma}(\gamma)$
$w = u\, \mathtt{a}\, \mathtt{b}^n\, \mathtt{a}\, v$,
n is even $\Rightarrow w \in L_{\mathrm{E},\Sigma}(\beta)$.

Union of General Pattern Languages

Example for "$\mathsf{E} \cup \mathsf{E} = \mathsf{E}$" and alphabet size 2:

$\Sigma = \{\mathtt{a}, \mathtt{b}\}$,
$\alpha = x_1 \mathtt{a} x_2 \mathtt{b} x_2 \mathtt{a} x_3$,
$\beta = x_1 \mathtt{a} x_2 \mathtt{b}\mathtt{b} x_2 \mathtt{a} x_3$,
$\gamma = x_1 \mathtt{a} x_2 \mathtt{b} x_3 \mathtt{a} x_4$.

$L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma)$,
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$L_{\mathsf{E},\Sigma}(\beta) \not\subseteq L_{\mathsf{E},\Sigma}(\alpha)$.

Proof sketch:

$L_{\mathsf{E},\Sigma}(\alpha) \subseteq L_{\mathsf{E},\Sigma}(\gamma)$ and
$L_{\mathsf{E},\Sigma}(\beta) \subseteq L_{\mathsf{E},\Sigma}(\gamma)$ is obvious.

Let $w \in L_{\mathsf{E},\Sigma}(\gamma)$
$w = u\, \mathtt{a}\, \mathtt{b}^n\, \mathtt{a}\, v$,
n is even $\Rightarrow w \in L_{\mathsf{E},\Sigma}(\beta)$.
n is odd $\Rightarrow w \in L_{\mathsf{E},\Sigma}(\alpha)$.

Union of General Pattern Languages

Example for "$\mathsf{E} \cup \mathsf{E} = \mathsf{E}$" and alphabet size 3:

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$,
$\alpha = x_1 \mathtt{a} x_2 x_3^6 x_4^3 x_5^6 x_6 \mathtt{b} x_7 \mathtt{a} x_2 x_8^{12} x_4^6 x_9^{12} x_6 \mathtt{b} x_{10}$,
$\beta = x_1 \mathtt{a} x_2 x_3^6 x_4^2 x_5^5 x_6^6 x_7 \mathtt{b} x_8 \mathtt{a} x_2 x_9^{12} x_4^4 x_5^{10} x_{10}^{12} x_7 \mathtt{b} x_{11}$,
$\gamma = x_1 \mathtt{a} x_2 x_3^6 x_4^2 x_5^3 x_6^6 x_7 \mathtt{b} x_8 \mathtt{a} x_2 x_9^{12} x_4^4 x_5^6 x_{10}^{12} x_7 \mathtt{b} x_{11}$.

$L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma)$,
$L_{\mathsf{E},\Sigma}(\alpha) \not\subseteq L_{\mathsf{E},\Sigma}(\beta)$,
$L_{\mathsf{E},\Sigma}(\beta) \not\subseteq L_{\mathsf{E},\Sigma}(\alpha)$.

Union of General Pattern Languages

Example for "$E \cup E = E$" and alphabet size 4:

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$,

$$\begin{aligned}
\alpha &:= x_1 \mathtt{a} x_2 x_3^2 x_4^2 x_5^2 x_6 \mathtt{b} x_7 \mathtt{a} x_2 x_8^2 x_4^2 x_9^2 x_6 \mathtt{b} \\
&\quad x_{10} \mathtt{c} x_{11} x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{16} \mathtt{d} x_{17} \mathtt{c} x_{11} x_{18}^2 x_{13}^2 x_{14}^2 x_{19}^2 x_{16} \mathtt{d} \\
&\quad x_{20} x_{13}^2 x_{14}^2 x_{13}^2 x_{14}^2 x_{13}^2 x_{14}^2 x_{21} x_4^6, \\
\beta &:= x_1 \mathtt{a} x_2 x_3^2 x_4^2 x_5^2 x_6^2 x_7 \mathtt{b} x_8 \mathtt{a} x_2 x_9^2 x_4^2 x_5^2 x_{10}^2 x_7 \mathtt{b} \\
&\quad x_{11} \mathtt{c} x_{12} x_{13}^2 x_{14}^2 x_{15}^2 x_{16} \mathtt{d} x_{17} \mathtt{c} x_{12} x_{18}^2 x_{14}^2 x_{19}^2 x_{16} \mathtt{d} \\
&\quad x_{20} x_{14}^6 x_{21} x_4^2 x_5^2 x_4^2 x_5^2 x_4^2 x_5^2 \text{ and} \\
\gamma &:= x_1 \mathtt{a} x_2 x_3^2 x_4^2 x_5^2 x_6^2 x_7 \mathtt{b} x_8 \mathtt{a} x_2 x_9^2 x_4^2 x_5^2 x_{10}^2 x_7 \mathtt{b} \\
&\quad x_{11} \mathtt{c} x_{12} x_{13}^2 x_{14}^2 x_{15}^2 x_{16}^2 x_{17} \mathtt{d} x_{18} \mathtt{c} x_{12} x_{19}^2 x_{14}^2 x_{15}^2 x_{20}^2 x_{17} \mathtt{d} \\
&\quad x_{21} x_{14}^2 x_{15}^2 x_{14}^2 x_{15}^2 x_{14}^2 x_{15}^2 x_{22} x_4^2 x_5^2 x_4^2 x_5^2 x_4^2 x_5^2 .
\end{aligned}$$

$L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma)$, $L_{\mathsf{E},\Sigma}(\alpha) \not\subseteq L_{\mathsf{E},\Sigma}(\beta)$, $L_{\mathsf{E},\Sigma}(\beta) \not\subseteq L_{\mathsf{E},\Sigma}(\alpha)$.

Necessary Condition for $E \cup E = E$

$\alpha = \alpha_0 u_1 \alpha_1 u_2 \alpha_2 \ldots \alpha_{n-1} u_n,$
$\beta = \beta_0 v_1 \beta_1 v_2 \beta_2 \ldots \beta_{m-1} v_m,$
$\gamma = \gamma_0 w_1 \gamma_1 w_2 \gamma_2 \ldots \gamma_{m-1} w_k,$
$\alpha_i, \beta_i, \gamma_i \in X^+,\ u_i, v_i, w_i \in \Sigma^+.$

Necessary Condition for $\mathsf{E} \cup \mathsf{E} = \mathsf{E}$

$\alpha = \alpha_0 u_1 \alpha_1 u_2 \alpha_2 \ldots \alpha_{n-1} u_n$,
$\beta = \beta_0 v_1 \beta_1 v_2 \beta_2 \ldots \beta_{m-1} v_m$,
$\gamma = \gamma_0 w_1 \gamma_1 w_2 \gamma_2 \ldots \gamma_{m-1} w_k$,
$\alpha_i, \beta_i, \gamma_i \in X^+$, $u_i, v_i, w_i \in \Sigma^+$.

$$L_{\mathsf{E},\Sigma}(\alpha) \cup L_{\mathsf{E},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma)$$

$$\Longrightarrow$$

$w_0 w_1 \ldots w_k = u_0 u_1 \ldots u_k$ and $w_0 w_1 \ldots w_k$ subsequence of $v_0 v_1 \ldots v_k$ or
$w_0 w_1 \ldots w_k = v_0 v_1 \ldots v_k$ and $w_0 w_1 \ldots w_k$ subsequence of $u_0 u_1 \ldots u_k$

Necessary Condition for $\mathsf{NE} \cup \mathsf{NE} = \mathsf{NE}$

Let $\{\mathtt{a}, \mathtt{b}\} \subseteq \Sigma$, let α, β and γ be patterns with neither

- $L_{\mathsf{NE},\Sigma}(\alpha) \subseteq L_{\mathsf{NE},\Sigma}(\beta)$, $\beta = \gamma$ nor
- $L_{\mathsf{NE},\Sigma}(\beta) \subseteq L_{\mathsf{NE},\Sigma}(\alpha)$, $\alpha = \gamma$.

Necessary Condition for NE $\cup$ NE $=$ NE

Let $\{\mathtt{a}, \mathtt{b}\} \subseteq \Sigma$, let α, β and γ be patterns with neither

- $L_{\mathrm{NE},\Sigma}(\alpha) \subseteq L_{\mathrm{NE},\Sigma}(\beta)$, $\beta = \gamma$ nor
- $L_{\mathrm{NE},\Sigma}(\beta) \subseteq L_{\mathrm{NE},\Sigma}(\alpha)$, $\alpha = \gamma$.

$$L_{\mathrm{NE},\Sigma}(\alpha) \cup L_{\mathrm{NE},\Sigma}(\beta) = L_{\mathrm{NE},\Sigma}(\gamma)$$

$$\Longrightarrow$$

$$|\Sigma| = 2$$

$$\alpha = \delta_0 \, \mathtt{a} \, \delta_1 \, \mathtt{a} \, \delta_2 \ldots \delta_{m-1} \, \mathtt{a} \, \delta_m \, ,$$

$$\beta = \delta_0 \, \mathtt{b} \, \delta_1 \, \mathtt{b} \, \delta_2 \ldots \delta_{m-1} \, \mathtt{b} \, \delta_m \, ,$$

$$\gamma = \delta_0 \, x \, \delta_1 \, x \, \delta_2 \ldots \delta_{m-1} \, x \, \delta_m \, ,$$

where $m \geq 1$, $\delta_i \in (X \cup \Sigma)^*$, $0 \leq i \leq m$.

Characterisations for $\mathsf{NE} \cup \mathsf{NE} = \mathsf{E}$

Let $|\Sigma| \geq 2$, let α, β and γ be patterns.

Characterisations for $\mathsf{NE} \cup \mathsf{NE} = \mathsf{E}$

Let $|\Sigma| \geq 2$, let α, β and γ be patterns.

$$L_{\mathsf{NE},\Sigma}(\alpha) \cup L_{\mathsf{NE},\Sigma}(\beta) = L_{\mathsf{E},\Sigma}(\gamma)$$

$$\Longleftrightarrow$$

$$\alpha = u_1\, u_2 \ldots u_{m+1} \in \Sigma^+ \text{ and } \beta = \gamma = u_1\, x^{j_1}\, u_2\, x^{j_2} \ldots x^{j_m}\, u_{m+1}, j_i \in \mathbb{N}_0.$$

Characterisations for $\mathsf{NE} \cup \mathsf{NE} = \mathsf{E}$

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This corresponds to the canonical way of expressing E-pattern languages by unions of NE-pattern languages.

Characterisations for $\mathsf{E} \cup \mathsf{E} = \mathsf{NE}$

Let $\{\mathtt{a}_1, \mathtt{a}_2, \ldots, \mathtt{a}_\ell\} \subseteq \Sigma$, $\ell \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, γ be patterns with $L_{\mathsf{E},\Sigma}(\alpha_i) \neq L_{\mathsf{E},\Sigma}(\alpha_j)$.

Characterisations for $\mathsf{E} \cup \mathsf{E} = \mathsf{NE}$

Let $\{a_1, a_2, \ldots, a_\ell\} \subseteq \Sigma$, $\ell \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, γ be patterns with $L_{\mathsf{E},\Sigma}(\alpha_i) \neq L_{\mathsf{E},\Sigma}(\alpha_j)$.

$$\bigcup_{i=1}^{\ell} L_{\mathsf{E},\Sigma}(\alpha_i) = L_{\mathsf{NE},\Sigma}(\gamma)$$

$$\Longleftrightarrow$$

$$\Sigma = \{a_1, a_2, \ldots, a_\ell\}$$

$$\gamma = u_1\, x\, u_2\, x\, u_3 \ldots u_k\, x\, u_{k+1}$$

$$\alpha_i = u_1\, \alpha_i'\, a_i\, \alpha_i''\, u_2\, \alpha_i'\, a_i\, \alpha_i''\, u_3 \ldots u_k\, \alpha_i'\, a_i\, \alpha_i''\, u_{k+1}$$

$\alpha_i', \alpha_i'' \in X^*$ and,

there exists a variable y_i with exactly one occurrence in $\alpha_i'\, a_i\, \alpha_i''$.

Thank you very much for your attention.