Pumping lemma and Ogden lemma for displacement context-free grammars

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    - Limited number of cross-serial dependencies.

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- MIX is not a 2-wMCFL (equivalently, not a tree-adjoining language) (Kanazawa, Salvati, 2012).
- Probably, it is not a k-wMCFL for any k (Kanazawa-Salvati conjecture, 2012). But how to prove it?

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- This operations generate the algebra of well-nested operations on tuples.
- Instead of tuples, gapped strings may be used.

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- $C[x_1, \ldots, x_t]$ — a multicontext, $\alpha_1, \ldots, \alpha_t$ — terms, s.t $\forall j \; (rk(\alpha_j) = \mathrm{rk}(x_j))$, then $C[\alpha_1, \ldots, \alpha_t]$ — substitution of all x_j-s by α_j-s.

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- $L_G(A) = \{\nu(\alpha) \mid A \vdash_G \alpha,\ \alpha \in GrTm_k\},\quad L(G) = L_G(S)$.

Example

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$((\langle aba, ba, ba \rangle) \odot_1 a) \odot_1 a = abaabaaba.$

Chomsky normal form I

Two grammars are equivalent if they generate the same language.

Theorem (Chomsky normal form)

Every k-DCFG is equivalent to a k-DCFG $G = \langle N, \Sigma, P, S \rangle$ with the rules only of the form:

1. $A \to B \cdot C$, *where* $A \in N$, $B, C \in N - \{S\}$,
2. $A \to B \odot_j C$, *where* $j \leqslant k$, $A \in N$, $B, C \in N - \{S\}$,
3. $A \to a$ *or* $A \to \langle \varepsilon, \varepsilon \rangle$, *where* $A \in N$, $a \in \Sigma$,
4. $S \to \varepsilon$.

Chomsky normal form II

Example

The grammar
$G'_2 = \langle \{S, T_2, U_1, V_1, U_2, V_2, W_2, X_2, A_1, B_1, A, B\}, \{a, b\}, P, S\rangle$
with the following set of rules generates the language
$\{www \mid w \in \Sigma^+\}$.

$$
\begin{array}{rcl@{\qquad}rcl}
S & \to & U_1 \odot_1 A & S & \to & V_1 \odot_1 B \\
U_1 & \to & U_2 \odot_1 A & V_1 & \to & V_2 \odot_1 B \\
U_2 & \to & A \cdot T & V_2 & \to & B \cdot T \\
T_2 & \to & W_2 \odot_2 A_1 & T_2 & \to & X_2 \odot_2 B_1 \\
W_2 & \to & U_2 \odot_1 A_1 & X_2 & \to & V_2 \odot B_1 \\
U_2 & \to & \langle a, \varepsilon, \varepsilon\rangle & V_2 & \to & \langle b, \varepsilon, \varepsilon\rangle \\
A_1 & \to & \langle a, \varepsilon\rangle & B_1 & \to & \langle b, \varepsilon\rangle \\
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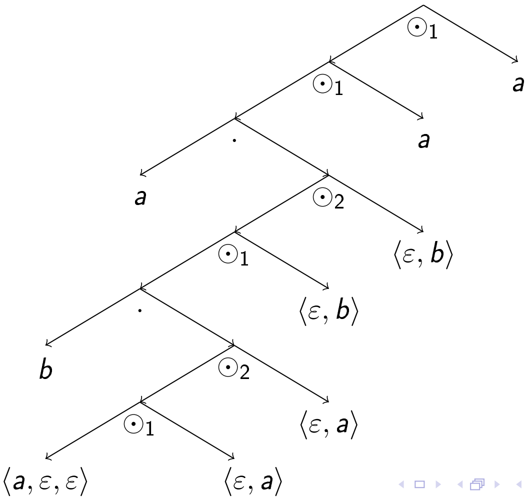
$$

Derivation tree

The derivation tree of the word *abaabaaba* in G_2':



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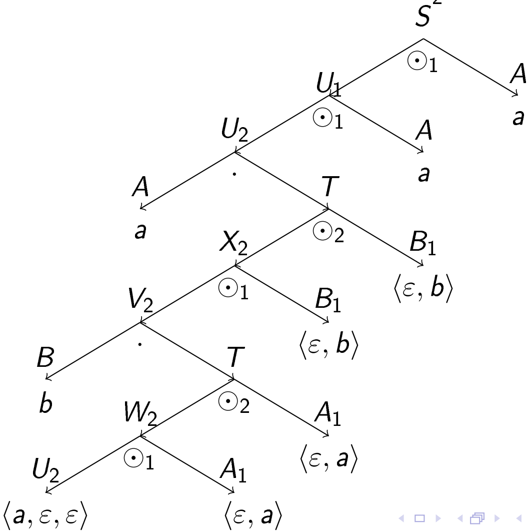

Properties of trees and terms

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- If $\alpha = C[\beta]$ for some ground context C and $rk(\beta) = l$, then there are tuples $s_1, s_2, u_1, \dots, u_l$ such that $\alpha = s_1(\beta \otimes (u_1, \dots, u_l))s_t$. ($\otimes$ denotes simultaneous replacement of all the separators)

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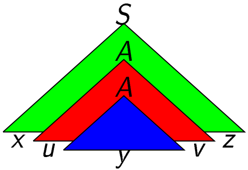
Lemma

If $\alpha = C[\beta]$ and β is a direct subterm of α, then there are words $s_1, s_2, u_1, v_1, \ldots, u_l, v_l$, s.t. $\alpha = s_1(\beta \otimes (\langle u_1, v_1 \rangle, \ldots, \langle u_l, v_l \rangle))s_2$.

Pumping lemma: reminder



$w = xuyvz$

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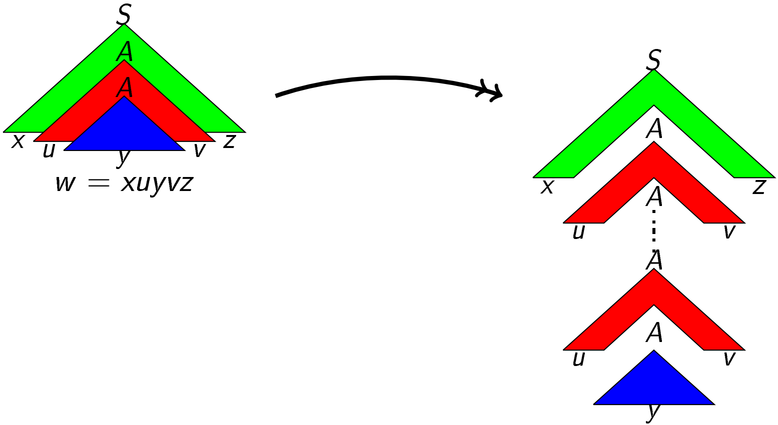

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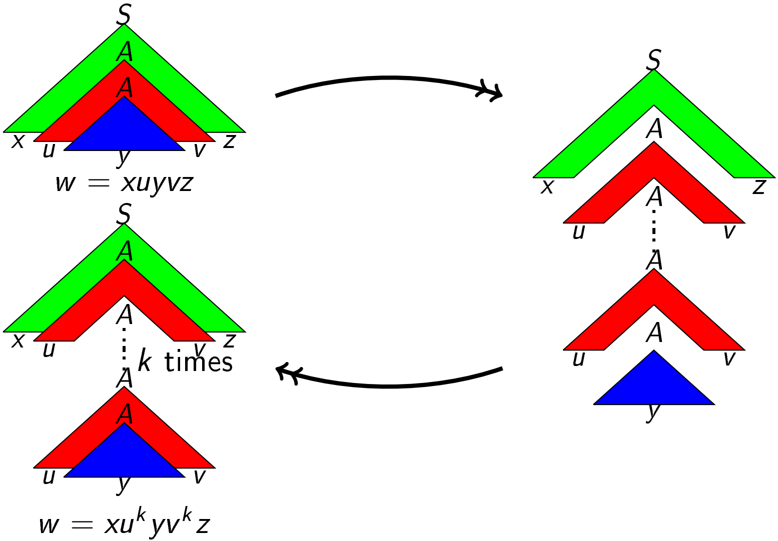

Problems for DCFGs

For any recursive terminal A of a CFG G there are words u, v s.t. $uv \neq \varepsilon$ and $A \vdash_G y$ implies $A \vdash_G uyv$.

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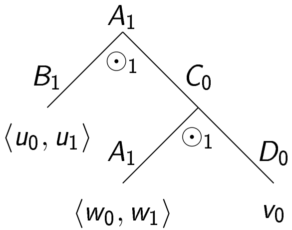

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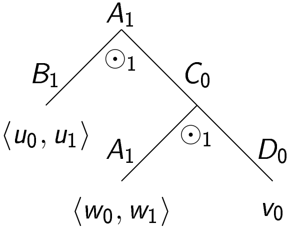

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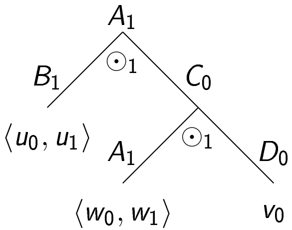

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That's because the foot node of the pumped subtree is not a direct descendant of its root.

Equivalence of terms

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Lemma

Every l-essential term α is equivalent to some l-correct β.

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- A term is l-redundant if it is l-essential, but not l-correct.
- Let N_l denote the number of nonterminals of rank l.
- A grammar G is l-duplicated if for any derivable rule $A \to \alpha$ with l-redundant α with $depth(\alpha) \leqslant N_l + 1$ there is a derivable rule $A \to \alpha'$, where $\alpha \sim \alpha'$ and $\alpha \in \mathrm{Tm}_l$. <4->

Lemma

For any k-DCFG G and any $l \leqslant k$ there is an equivalent l-duplicated k-DCFG G' with the same set of nonterminals of rank l and greater.

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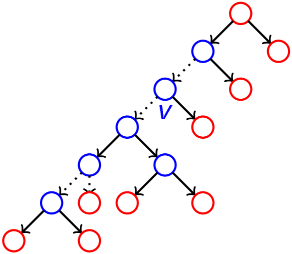

blue — the same rank as v
red — rank, diverse from v

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A derivation tree is m-compact, if for any node v of any rank l there is a path from l to an element of an l' - matreshka with $l' \geqslant l$ such that its length is not greater than l and all the nodes on it are of rank l or greater.

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Definition

A vicinity of a node v if the largest well-formed subtree containing T such that all its internal nodes have the same rank as v.

Definition

A derivation tree is m-compact, if for any node v of any rank l there is a path from l to an element of an l' - matreshka with $l' \geqslant l$ such that its length is not greater than l and all the nodes on it are of rank l or greater.

Lemma

For any k-DCFG G for some m there is an equivalent m-compact k-DCFG.

Sketch of the proof

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- If it contains some node of greater rank: use induction hypothesis.
- Otherwise it is l-redundant — replace the subtree by an equivalent one, which is l-correct.

Pumping lemma

Theorem (Pumping lemma for k-DCFGs)

For any k-DCFG G there is a number n such that for any word $w \in L(G)$, such that $|w| \geqslant n$, there is a decomposition $w = s_0 u_1 x_1 v_1 s_1 \ldots u_k x_k v_k s_k$, satisfying:

1. $\sum_{i=1}^{k} |u_i x_i v_i| \leqslant n, \quad u_1 v_1 \ldots u_k v_k \neq \varepsilon,$
2. *For any* $m \in \mathbb{N}$ $s_0 u_1^m x_1 v_1^m s_1 \ldots u_k^m x_k v_k^m s_k \in L(G)$.

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Theorem (Ogden lemma for 1-DCFGs (Palis, Shende, 1995))

For any TAG G there is a number n such that for any word winL(G) with at least n marked positions, such that $|w| \geqslant n$, there is a decomposition $w = s_0 u_1 x_1 v_1 s_1 u_2 x_2 v_2 s_2$, satisfying:

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2. *There is at least one marked position in one of* u_1, v_1, u_2, v_2.
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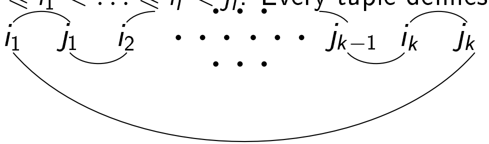

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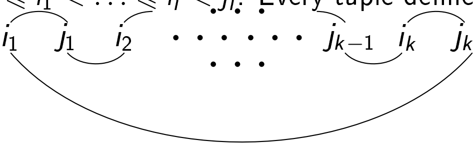

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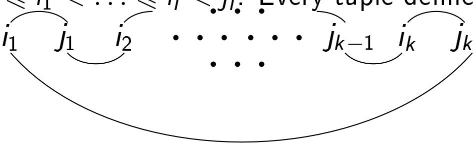

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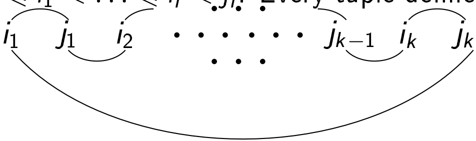

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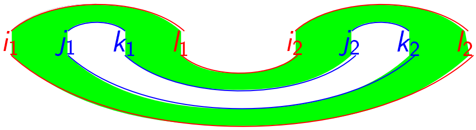

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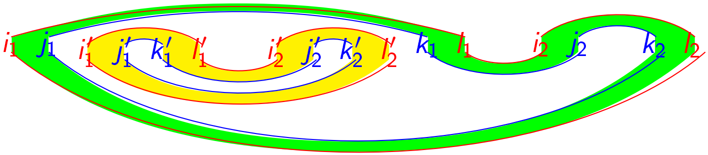

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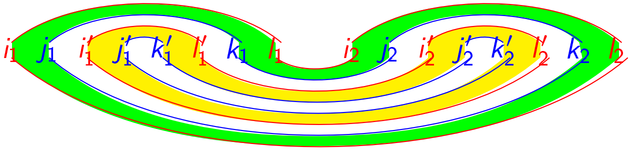

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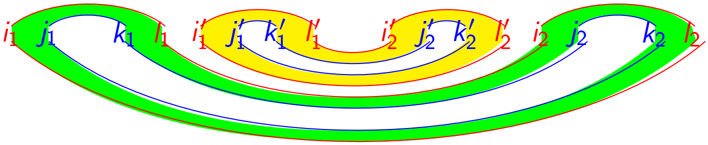

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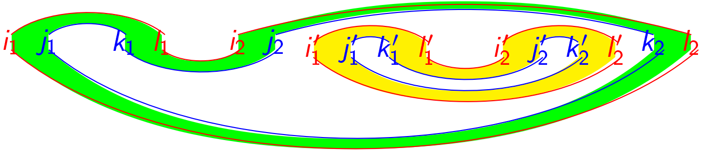

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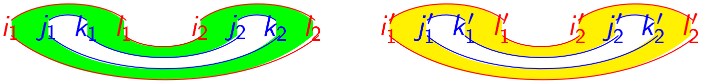

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- By analogous arguments, there is a $[1,4]$-pump.

4MIX language II

- $w = a^{m_1} b^{m_2} c^{m_3} d^{m_4} a^{n_1} b^{n_2} c^{n_3} d^{n_4}$,
  1. $\min(m_j, n_j) \geqslant t$, where t is the number from Ogden's lemma,
  2. $m_1 \geqslant (3M+1)(M+t)$, where $M = \max(m_2, m_4, n_3)$,
  3. $m_4 \geqslant (n_1+1)(n_1+t)$.

$$\underbrace{a \dots a}_{t} \quad \underset{M}{\longleftrightarrow} \quad \underbrace{a \dots a}_{t} \quad \underset{M}{\longleftrightarrow} \quad \dots \quad \underset{M}{\longleftrightarrow} \quad \underbrace{a \dots a}_{t}$$

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- If all the groups belong to different pumps, then there are at least $M+2$ non-intersecting $[1,2]$ pumps, but m_2 is too small. Again, there exists a $[1,3,6,8]$-pump.
- By analogous arguments, there is a $[1,4]$-pump.
- It should be a $[1,4,8]$-pump, which is impossible. Then $\mathrm{4MIX}$ is not a 2-DCFL.

MIX language

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MIX *is not a* 1-*DCFL.*

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- We prove for $\mathrm{MIX} \cap a^+b^+c^+b^+c^+a^+$, $w = a^{m_1}b^{m_2}c^{m_3}b^{n_2}c^{n_3}a^{n_1}$.

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- There is a $[2,3]$-group, which should be a $[1,2,3,6]$-group
- There is a [4]-group, but it could be only a $[1,2,4,6]$-group, which has no c-s. Contradiction.

MIX_p language

Theorem

$\mathrm{MIX}_p = \{w \in \{a,b,c\}^+ \mid |w|_a = |w|_b = |w|_c, \forall u \subseteq w |u|_a \geqslant |u|_b \geqslant |u|_c\}$ *is not a 1-DCFL.*

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- If the first is embracing the second, it is a [1, 2, 4, 6]-pump — impossible to combine them.

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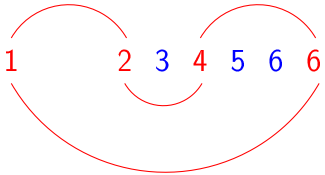

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- If the first is embracing the second, it is a [1, 2, 4, 6]-pump — impossible to combine them.
- If the second — symmetrically. Contradiction.

Future work

- Ogden's lemma for k-DCFGs (or any other technique to localize the pumps).

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- Apply this lemma to give counterexamples on higher levels of DCFG hierarchy.
- Kanazawa-Salvati conjecture (MIX is not a DCFL).

Thank you!

Спасибо за внимание!